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Christian Stelzer-Landauer

Approximation of Dirac Operators with Delta-Shell Potentials in the Norm Resolvent Sense

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Norm Resolvent Sense”, presented at Graz University of Technology,
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Abstract

The present thesis is devoted to the approximation of Dirac operators with δ-shell potentials supported on the boundary of a two or three-dimensional C^2-domain. These singular potentials are used as idealized replacements for potentials which are strongly localized in a neighbourhood of the support of the δ-shell potential and they often simplify the spectral analysis. To justify the usage of such potentials it is essential to prove that Dirac operators with δ-shell potentials can be approximated by Dirac operators with strongly localized potentials in a way which transfers the spectral properties. The most important contribution of this thesis is the establishment of conditions for the convergence of Dirac operators with strongly localized potentials in the norm resolvent sense. This type of convergence implies that the spectrum of the Dirac operator with δ-shell potential can be completely characterized by the spectra of the approximating operators and vice versa. In the special case of electrostatic and Lorentz scalar δ-shell potentials an explicit convergence condition is provided. Furthermore, counterexamples which imply the sharpness of this condition are also presented.

Zusammenfassung

Das Ziel dieser Dissertation ist es, Dirac Operatoren mit δ-Potentialen, welche auf dem Rand eines zwei- oder dreidimensionalen C^2-Gebietes definiert sind, zu approximieren. Derartige δ-Potentiale werden als Idealisierung von regulären Potentialen gesehen, welche stark in der Umgebung des Trägers des δ-Potentials lokalisert sind. Um eine Verwendung von solchen Potentialen rechtzufertigen, muss gezeigt werden, dass Dirac Operatoren mit δ-Potentialen durch Dirac Operatoren mit stark lokalisierten Potentialen auf eine Weise angenähert werden können, welche auch spektrale Eigenschaften überträgt. Der wichtigste Beitrag dieser Arbeit zur aktuellen Forschung ist die Angabe von Bedingungen für die Konvergenz im Normresolventensinn. Konvergenz in diesem Sinn impliziert, dass das Spektrum des Dirac Operators mit δ-Potential vollständig durch die Spektren der approximierenden Operatoren charakterisiert werden kann und umgekehrt dasselbe gilt. Für den Spezialfall von elektrostatischen und Lorentz-skalaren δ-Potentialen wird eine explizite Konvergenzbedinung angegeben. Durch passende Gegenbeispiele wird zusätzlich gezeigt, dass die Bedingung auch scharf ist.

Contents

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1 Introduction

1.1 Description of the problem

Differential operators coupled with singular potentials are frequently used in mathematical physics. Such singular potentials model regular potentials which have very large values in the vicinity of a set of measure zero and small values everywhere else. In contrast to regular potentials, they cannot be represented by functions and have to be described by distributions. Nonetheless, the spectral analysis of differential operators with singular potentials often simplifies substantially and may even reduce to an explicitly solvable problem.

To justify the replacement of regular potentials by singular potentials, approximation results are necessary, i.e. if H is a differential operator and S is a singular potential, one has to show that $H+S$ can be approximated by $H+V_\varepsilon$ as $\varepsilon \to 0$ in a suitable sense, where $(V_\varepsilon)_{\varepsilon>0}$ is a family of regular potentials which converges in the distributional sense to S. Here, "suitable sense" means in particular that the convergence should relate the spectra as well as the associated spectral projections of $H+S$ and $H+V_\varepsilon$. In the context of self-adjoint unbounded operators, appropriate notions of convergence are the strong resolvent convergence and the norm resolvent convergence, which means that $(H + V_\varepsilon - i)^{-1}$ converges to $(H + S - i)^{-1}$ as $\varepsilon \to 0$ in the strong sense or in the operator norm, respectively.

In this thesis we focus on the approximation of Dirac operators with δ-shell potentials. We start by explaining Dirac operators. They were introduced by Paul Dirac in 1929 and are used to describe spin 1/2 particles in a quantum mechanical framework. Moreover, in contrast to Schrödinger operators, they also comply with the theory of relativity. The free Dirac operator without any potential has the following form in natural units:

$$H = -i\sum_{j=1}^{\theta} \alpha_j \partial_j + m\beta, \qquad \operatorname{dom} H = H^1(\mathbb{R}^\theta;\mathbb{C}^N) \subset L^2(\mathbb{R}^\theta;\mathbb{C}^N). \tag{1.1}$$

Here, $\theta \in \{1,2,3\}$ denotes the space dimension, $m \in \mathbb{R}$ describes the mass of a particle and $\alpha_1,\dots,\alpha_\theta,\beta \in \mathbb{C}^{N\times N}$, $N = 2\lceil\frac{\theta}{2}\rceil$, are the Dirac matrices introduced in Definition 3.1 for $\theta \in \{2,3\}$. For $\theta \in \{1,2\}$ one has $\alpha_j = \sigma_j$, $j \in \{1,\dots,\theta\}$, and

$\beta = \sigma_3$, with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Adding a symmetric matrix-valued potential S to H allows one to also model the influence of external fields, e.g. electrostatic, Lorentz scalar or magnetic fields. In this thesis we are interested in the case $S = V\delta_\Sigma$, where δ_Σ is the singular δ-shell potential supported on a C^2-smooth hypersurface $\Sigma \subset \mathbb{R}^\theta$ and V is a symmetric matrix-valued function on Σ. If $V = \eta I_N + \tau\beta + \lambda i(\alpha \cdot \nu)\beta + \omega(\alpha \cdot \nu)$, where ν is the unit normal vector on Σ and $\alpha \cdot \nu = \sum_{j=1}^{\theta} \alpha_j \nu_j$, we call the scalar functions η, τ, λ and ω the electrostatic, Lorentz scalar, anomalous magnetic and magnetic interaction strengths, respectively.
The main aim of this thesis is to study the norm resolvent convergence of the operators $H_{V_\varepsilon} = H + V_\varepsilon$, where $(V_\varepsilon)_{\varepsilon>0}$ is a family of regular potentials which converges for $\varepsilon \to 0$ in the distributional sense to $V\delta_\Sigma$.

1.2 State of the art

For Schrödinger operators, which are the nonrelativistic counterparts of Dirac operators, the literature regarding such approximation results is extensive. It is well-known that in the one-dimensional setting, where Σ is either a single point or a countable set of points, Schrödinger operators with δ-potentials can be approximated in the norm resolvent sense by Schrödinger operators with strongly localized potentials; see for instance [1] and the references therein. In two and three dimensions this problem was also considered for various choices of Σ; see [2, 30, 31, 58, 68]. Furthermore, in [7] norm resolvent convergence was proven for a general class of C^2-smooth hypersurfaces in the multidimensional setting.

Approximation problems for one-dimensional Dirac operators with $\Sigma = \{0\}$ were first considered in 1989 by Šeba in [67]. He investigated Dirac operators with potentials of the form $V_\varepsilon = V h_\varepsilon$, where $V = \eta I_2$, $\eta \in \mathbb{R}$, or $V = \tau\beta$, $\tau \in \mathbb{R}$, and $(h_\varepsilon)_{\varepsilon>0}$ is a suitable family of functions converging for $\varepsilon \to 0$ in the distributional sense to δ_0, which is the δ-potential supported in $\{0\}$. In this setting he was able to show that H_{V_ε} converges in the norm resolvent sense to the operator $H_{\widetilde{V}\delta_0}$. Here, $\widetilde{V} = \widetilde{\eta} I_2$ or $\widetilde{V} = \widetilde{\tau}\beta$, where $\widetilde{\eta}$ and $\widetilde{\tau}$ are rescaled interaction strengths which depend nonlinearly on η and τ, respectively. This rescaling does not appear in the case of Schrödinger operators and had already been observed a few years prior in various physics papers, see e.g. [22, 52, 53], when comparing the solutions of the Dirac eigenvalue equation with δ-potentials and strongly localized potentials. Furthermore, Šeba and the authors of [52, 53] related this phenomenon to Klein's paradox. In the nineties Hughes showed in [40, 41, 42] that H_{Vh_ε} converges in the strong resolvent sense to $H_{\widetilde{V}\delta_0}$ for self-adjoint matrices $V \in \mathbb{C}^{2\times 2}$. Moreover, she was also able to find an explicit formula for the rescaling of V to $\widetilde{V}$; cf. [42, Theorem 1 and Theorem 2]. In our terminology this formula is given by

$$\widetilde{V} = 2\alpha_1 \sin\Big(\frac{\alpha_1 V}{2}\Big) \cos\Big(\frac{\alpha_1 V}{2}\Big)^{-1} \tag{1.2}$$

provided that $\cos\big(\frac{\alpha_1 V}{2}\big)$ is an invertible matrix. Finally, in 2020, Tušek extended in [72] the works of Šeba and Hughes by proving norm resolvent convergence in the one-dimensional setting for a large class of self-adjoint interaction matrices $V \in \mathbb{C}^{2\times 2}$.

In the multidimensional setting the literature is less complete than in the one-dimensional case and so far there only exist results on strong resolvent convergence. In this setting one defines based on a matrix-valued function V given on Σ, for $\varepsilon > 0$ a potential V_ε which is supported in an ε-neighbourhood of Σ and converges for $\varepsilon \to 0$ in the distributional sense to $V\delta_\Sigma$. In 2018 Mas and Pizzichillo considered this problem in three dimensions in [51], where Σ was assumed to be a compact C^2-surface. Inspired by the methods used in [7] for the approximation of Schrödinger operators with δ-shell potentials, they were able to show strong resolvent convergence in the case of purely electrostatic and purely Lorentz scalar interactions, if the interaction strengths satisfy a nonexplicit smallness condition. Moreover, they observed a similar rescaling of V to $\widetilde{V}$ as known from the one-dimensional counterpart. For the special case where Σ is the sphere, the same authors considered the convergence of the eigenvalues in [50]. Recently, the two-dimensional case with Σ being a smooth closed curve was considered for the first time in [24] by Cassano, Lotoreichik, Mas and Tušek. In this paper the authors established strong resolvent convergence of H_{V_ε} to $H_{\widetilde{V}\delta_\Sigma}$ for interaction matrices of the type $V = \eta I_N + \tau\beta + \lambda i(\alpha \cdot \nu)\beta$ without any smallness assumption. Behrndt, Holzmann and Tušek showed an analogous statement in the case where Σ is a straight line in [18]. Furthermore, in [74], Zreik transferred the methods from [24] to the three-dimensional setting and showed that H_{V_ε} converges in the strong resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$ for combinations of electrostatic and Lorentz scalar interaction strengths.

A different approach to approximate Dirac operators with δ-shell potentials, which goes back to [23, 67], is via so-called nonlocal potentials; see also [34]. In one dimension such potentials are given by $\underline{V_\varepsilon} = (\cdot, h_\varepsilon)_{L^2(\mathbb{R})} V h_\varepsilon$ and they also converge in the distributional sense to $V\delta_0$. However, in contrast to the classical strongly localized potentials, $H_{\underline{V_\varepsilon}}$ converges in the norm resolvent sense to $H_{V\delta_0}$, i.e. no rescaling of the interaction matrix is necessary. Tušek and Heriban took up this idea in [35] and considered the norm resolvent convergence of Dirac operators with such potentials in the multidimensional case. It turned out that in the mentioned case no rescaling is necessary either, but in contrast to the one-dimensional case, the limit operator is not a Dirac operator with a local δ-shell potential but rather a Dirac operator with a so-called nonlocal δ-shell potential.

1.3 Rigorous definition of various objects

Before we state and discuss the main results of this thesis, let us fix some necessary notations. We assume that $\Omega_+ \subset \mathbb{R}^\theta$, $\theta \in \{2,3\}$, is a possibly unbounded C^2-smooth domain (according to Definition 2.1) and we set $\Sigma := \partial\Omega_+$ and $\Omega_- := \mathbb{R}^\theta \setminus \overline{\Omega_+}$. Moreover, we denote the unit outward normal vector field of Ω_+ by ν. For a function $u : \mathbb{R}^\theta \to \mathbb{C}^N$ we write $u_\pm := u \upharpoonright \Omega_\pm$ and we denote the Dirichlet trace operator by $\mathbf{t}_\Sigma^\pm : H^1(\Omega_\pm;\mathbb{C}^N) \to H^{1/2}(\Sigma;\mathbb{C}^N)$, where H^s are the L^2-based Sobolev spaces. Recall that $\alpha_1,\dots,\alpha_\theta,\beta \in \mathbb{C}^{N\times N}$ are the Dirac matrices defined in Definition 3.1. To shorten notation, we make use of the abbreviations

$$\alpha\cdot\nabla := \sum_{j=1}^{\theta}\alpha_j\partial_j \quad \text{and} \quad \alpha\cdot x := \sum_{j=1}^{\theta}\alpha_j x_j, \quad x=(x_1,\dots,x_\theta)\in\mathbb{C}^\theta.$$

For $m\in\mathbb{R}$ and $\widetilde{V}\in L^\infty(\Sigma;\mathbb{C}^{N\times N})$ such that $\widetilde{V}=\widetilde{V}^*$, i.e. $\widetilde{V}(x_\Sigma)=(\widetilde{V}(x_\Sigma))^*$ for a.e. $x_\Sigma\in\Sigma$, we introduce the operator

$$\begin{aligned} H_{\widetilde{V}\delta_\Sigma}u &:= (-i(\alpha\cdot\nabla)+m\beta)u_+ \oplus (-i(\alpha\cdot\nabla)+m\beta)u_-,\\ \operatorname{dom} H_{\widetilde{V}\delta_\Sigma} &:= \Big\{ u\in H^1(\Omega_+;\mathbb{C}^N)\oplus H^1(\Omega_-;\mathbb{C}^N):\\ &\qquad i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma^+u_+ - \boldsymbol{t}_\Sigma^-u_-) + \frac{\widetilde{V}}{2}(\boldsymbol{t}_\Sigma^+u_+ + \boldsymbol{t}_\Sigma^-u_-) = 0\Big\} \subset L^2(\mathbb{R}^\theta;\mathbb{C}^N). \end{aligned} \tag{1.3}$$

This is a rigorous realization of the formal operator

$$H+\widetilde{V}\delta_\Sigma = -i(\alpha\cdot\nabla)+m\beta+\widetilde{V}\delta_\Sigma,$$

which was studied under various assumptions on $\widetilde{V}$ and the interaction support Σ in [4, 5, 8, 9, 13, 16, 18, 19, 24, 59, 60]. The operator exhibits different behaviours depending on the properties of $\widetilde{V}$. To illustrate this fact, let us consider $\widetilde{V}=\widetilde{\eta}I_N+\widetilde{\tau}\beta$ with $\widetilde{\eta},\widetilde{\tau}\in\mathbb{R}$. It is well-known that $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint if $\widetilde{d}=\widetilde{\eta}^2-\widetilde{\tau}^2\neq 4$. This case is referred to as the *noncritical* case. In the *critical* case, i.e. $\widetilde{d}=4$, $H_{\widetilde{V}\delta_\Sigma}$ is not closed and only essentially self-adjoint; see [12, 13, 19, 56]. Not only this, by [12, 13, 17, 18, 19, 20] the spectral properties also change drastically in the critical case. We conclude the discussion of $H_{\widetilde{V}\delta_\Sigma}$ by mentioning that if $\widetilde{d}=\widetilde{\eta}^2-\widetilde{\tau}^2=-4$, then $H_{\widetilde{V}\delta_\Sigma}$ splits into the orthogonal sum of two operators acting in $L^2(\Omega_+;\mathbb{C}^N)$ and $L^2(\Omega_-;\mathbb{C}^N)$; cf. Proposition 3.15 (ii). This implies that the particle described by

$H_{\widetilde{V}\delta_\Sigma}$ cannot cross Σ and thus stays confined either in Ω_+ or in Ω_-. Hence, we say that $H_{\widetilde{V}\delta_\Sigma}$ induces confinement in this case.

It is the main goal of the present thesis to show that $H_{\widetilde{V}\delta_\Sigma}$ can be approximated in the norm resolvent sense by Dirac operators with strongly localized potentials. To introduce the latter operators, we define the map

$$\iota : \Sigma \times \mathbb{R} \to \mathbb{R}^\theta, \quad \iota(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma), \quad (x_\Sigma, t) \in \Sigma \times \mathbb{R},$$

and for $\varepsilon \in (0,\infty)$ we set $\Omega_\varepsilon := \iota(\Sigma \times (-\varepsilon, \varepsilon))$, which is the so-called *tubular neighbourhood* of Σ. Furthermore, for $\varepsilon_{\text{tub}} > 0$ sufficiently small the map $\iota \upharpoonright \Sigma \times (-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}})$ is injective; cf. Proposition 2.12. To define the above mentioned strongly localized potentials, we choose

$$q \in L^\infty((-1,1);\mathbb{R}) \quad \text{with} \quad \int_{-1}^{1} q(s)\,ds = 1$$

and

$$V \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N}) \quad \text{such that} \quad V = V^*,$$

where W^1_∞ denotes the first order L^∞-based Sobolev space. Since $\iota \upharpoonright \mathbb{R} \times (-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}})$ is injective, we can define for $\varepsilon \in (0, \varepsilon_{\text{tub}})$

$$V_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} V(x_\Sigma) q\left(\frac{t}{\varepsilon}\right), & x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & x \notin \Omega_\varepsilon, \end{cases} \tag{1.4}$$

and for $m \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_{\text{tub}})$ the operator

$$H_{V_\varepsilon} u := -i(\alpha \cdot \nabla)u + m\beta u + V_\varepsilon u, \quad \operatorname{dom} H_{V_\varepsilon} := H^1(\mathbb{R}^\theta; \mathbb{C}^N).$$

Note that $H_{V_\varepsilon} = H + V_\varepsilon$ is self-adjoint in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ as $V_\varepsilon = V_\varepsilon^* \in L^\infty(\mathbb{R}^\theta; \mathbb{C}^{N\times N})$ and the free Dirac operator H from (1.1) is self-adjoint; cf. Proposition 3.3. Moreover, the sequence V_ε converges to $V\delta_\Sigma$ as $\varepsilon \to 0$ in the sense of distributions by construction.

Recall from Section 1.2 that the expected limit operator is not $H_{V\delta_\Sigma}$ but rather $H_{\widetilde{V}\delta_\Sigma}$, where V has been rescaled to $\widetilde{V}$. Provided that $\cos\left(\frac{(\alpha\cdot\nu)V}{2}\right)^{-1} \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N})$, the rescaling is given by

$$\widetilde{V} = 2(\alpha\cdot\nu)\sin\left(\tfrac{(\alpha\cdot\nu)V}{2}\right)\cos\left(\tfrac{(\alpha\cdot\nu)V}{2}\right)^{-1}, \tag{1.5}$$

where analytic functions of matrices are defined via the corresponding power series; cf. (1.2) for the one-dimensional counterpart of this formula. If $\eta, \tau \in W^1_\infty(\Sigma;\mathbb{R})$ and $V = \eta I_N + \tau\beta$, then the rescaling can be simplified to

$$\widetilde{V} = \frac{2\tan\left(\frac{\sqrt{d}}{2}\right)}{\sqrt{d}} V, \qquad d = \eta^2 - \tau^2.$$

In particular,

$$\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta \quad \text{with} \quad (\widetilde{\eta}, \widetilde{\tau}) = \frac{2\tan\left(\frac{\sqrt{d}}{2}\right)}{\sqrt{d}}(\eta, \tau) \tag{1.6}$$

in this special case, which is prevalent in literature; see [18, 24, 74] for analogous rescalings.

1.4 New results and structure of this thesis

Having established some necessary notations in Section 1.3, we are now in the position to discuss the main new contributions of this thesis. The results, which are partially included in the preprints [14, 15], are presented in a manner that also elucidates the structure of this dissertation. After the preliminary Chapters 2–3, we find in Chapter 4 abstract conditions which guarantee the norm resolvent convergence of H_{V_ε} to $H_{\widetilde{V}\delta_\Sigma}$ for $\varepsilon \to 0$, where V_ε is the strongly localized potential based on the interaction matrix V from (1.4) and $\widetilde{V} = \widetilde{V}(V)$ is the rescaled interaction matrix given by (1.5); see Theorem 4.15. In Corollary 4.16 we apply this theorem and show that if $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small, then the operator $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and H_{V_ε} converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$ for $\varepsilon \to 0$. Theorem 4.15 extends the current literature in the following three aspects:

(i) Instead of strong resolvent convergence, we prove the norm resolvent convergence of the approximating family, which has not been established in the multi-dimensional situation so far. This type of convergence ensures that the spectrum of the limit operator $H_{\widetilde{V}\delta_\Sigma}$ can be completely characterized by the spectra of the approximating operators and it also implies the convergence of the related spectral projections.

(ii) Instead of bounded curves in $\mathbb{R}^2$ or bounded surfaces in $\mathbb{R}^3$, we treat a general class of bounded and unbounded interaction supports Σ which we call *special C^2-surfaces*. This class of surfaces can be described by finitely many rotated graphs of C^2-functions with bounded derivatives; see Definition 2.1. In particular, this class includes graphs of C^2-functions with bounded derivatives and boundaries of bounded C^2-domains.

(iii) Instead of considering only electrostatic, Lorentz scalar, and anomalous magnetic interactions (which can be described by three real-valued functions), we allow general symmetric 2×2 or 4×4 matrix-valued functions as interaction strengths in dimensions two or three, respectively, and provide an explicit formula for rescaling when passing to the limit.

The key idea for the improvement from strong resolvent convergence to norm resolvent convergence mentioned in (i) is the following: We factorize the resolvent differences of the free Dirac operator and Dirac operators with strongly localized potentials and instead of viewing the individual factors as operators in L^2-spaces, we study their convergence properties as operators between various (Bochner-)Sobolev spaces.

In Chapter 5 we focus on interaction matrices having the form $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, where $C_b^1(\Sigma;\mathbb{R})$ is the set of all real-valued bounded C^1-smooth functions on Σ which have bounded first derivatives. The interaction strengths η and τ are used to model electrostatic and Lorentz scalar interactions, respectively, and they are the most common interaction types in the literature of Dirac operators with δ-shell potentials; see for instance [4, 9, 13, 19, 60]. In this setting we prove in Theorem 5.20 that the abstract conditions from Theorem 4.15 for norm resolvent convergence simplify to the explicit condition

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2. \tag{1.7}$$

Inspired by the last paragraph of [24, Section 8] we then add a strongly localized magnetic potential to H_{V_ε}; more precisely, we choose $V = \eta I_N + \tau\beta + \pi(\alpha\cdot\nu)$. It turns out that in this case H_{V_ε} also converges in the norm resolvent sense; see Theorem 5.21. However, by the specific choice of π as the magnetic interaction strength, the magnetic term disappears when rescaling. Hence, we end up with a limit operator $H_{\widetilde{V}\delta_\Sigma}$ which is again a Dirac operator with δ-shell potential and only electrostatic and Lorentz scalar interactions, where the rescaling of η and τ is different than in the case $V = \eta I_N + \tau\beta$. Using this result and Theorem 5.20 we can formulate Corollary 5.22, which states that every Dirac operator with a given δ-shell potential $\widetilde{V}\delta_\Sigma$, $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$, and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ fulfilling

$$\sup_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| < 4 \quad \text{or} \quad \inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| > 4 \tag{1.8}$$

can be approximated by a sequence of Dirac operators with strongly localized potentials. In the case of constant interaction strengths this is particularly interesting, as it implies that every Dirac operator with a δ-shell potential and constant electrostatic and Lorentz scalar interaction strengths satisfying $|\widetilde{d}| \neq 4$ can be approximated in the norm resolvent sense by Dirac operators with strongly localized potentials.

In Chapter 6 we show that the condition (1.7) is in fact optimal. We do this by providing suitable counterexamples. To discuss the counterexamples in more detail we assume $V = \eta I_N + \tau\beta$ with $\eta, \tau \in \mathbb{R}$. By (1.6) we have

$$\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4\tan\left(\tfrac{\sqrt{d}}{2}\right).$$

We distinguish between the critical ($\widetilde{d} = 4$) and the noncritical ($\widetilde{d} \neq 4$) case. Note that if $d < \frac{\pi^2}{4}$, i.e. (1.7) is fulfilled, then $\widetilde{d} < 4$ and hence $\widetilde{V}$ is noncritical. Now, let us assume that $d \geq \frac{\pi^2}{4}$, i.e. (1.7) is not fulfilled. If $\widetilde{d} = 4$, then $H_{\widetilde{V}\delta_\Sigma}$ is not closed and only essentially self-adjoint. Hence, H_{V_ε} cannot converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$. Moreover, as also the spectral properties change in the critical case, we are able to show in Theorem 6.1 that H_{V_ε} does not even converge to the closure of $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense if Σ is compact and C^∞-smooth. Furthermore, if $d \geq \frac{\pi^2}{4}$ and $\widetilde{d} \neq 4$ we show in Theorem 6.7 that H_{V_ε} does not converge to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense under the geometric assumption that Σ contains a flat part.

Chapter 7 is split into two parts. In the first part we approximate Dirac operators which induce confinement and in the second part we present various spectral implications of the approximation results. Let us start by explaining the first part in more detail. From (1.8) we conclude that the confinement case, i.e. $\widetilde{d} = -4$, is not included in our approximation results. To approximate Dirac operators with δ-shell potentials that induce confinement, we use the following approach: We choose $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ such that $\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < 0$. Moreover, we assume that $f : (0, \varepsilon_{\text{tub}}) \to (0, \infty)$ is a suitable scaling function with $f(\varepsilon) \to \infty$ for $\varepsilon \to 0$; the exact conditions are given in (7.2). Then, Theorem 7.4 states that $H_{f(\varepsilon)V_\varepsilon}$ converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$, where

$$\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta \qquad (\widetilde{\eta}, \widetilde{\tau}) = \frac{2}{\sqrt{|d|}}(\eta, \tau).$$

This immediately shows that $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = -4$, i.e. $\widetilde{V}$ induces indeed confinement. After considering the approximation of Dirac operators with δ-shell potentials which induce confinement, we deal in Section 7.2 with the discrete and essential spectrum of H_{V_ε}. In particular, we find conditions which guarantee the existence of discrete eigenvalues in various situations.

Finally, in Chapter 8, we introduce so-called *semilocal* potentials. Recall from Section 1.2 that one can use nonlocal potentials to approximate Dirac operators with δ-shell potentials without any rescaling in the one-dimensional setting. As already mentioned, this approach does not work for $\theta \in \{2, 3\}$. This leads us to the definition of semilocal potentials which allow approximations of Dirac operators with δ-shell potentials supported on Σ without any rescaling. For $V = V^* \in W^1_\infty(\Sigma, \mathbb{C}^{N\times N})$, $q \in L^\infty((-1,1); \mathbb{R})$ with $\int_{-1}^1 q(t)\,dt = 1$ and $\varepsilon \in (0, \varepsilon_{\text{tub}})$ we define

$$\underline{V_\varepsilon} : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N),$$

$$(\underline{V_\varepsilon} u)(x) := \begin{cases} \frac{1}{\varepsilon} V(x_\Sigma) q\big(\frac{t}{\varepsilon}\big) \int_{-1}^1 u(\iota(x_\Sigma, \varepsilon s)) q(s) \\ \qquad\qquad\qquad \cdot \det(I - s\varepsilon W(x_\Sigma))\,ds, & x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & x \notin \Omega_\varepsilon. \end{cases}$$

Here W denotes the Weingarten map associated with Σ and plays only a secondary role since it is scaled with ε. We call such potentials semilocal since they behave nonlocally with respect to the variable t in the normal direction of Σ and locally with respect to the variable $x_\Sigma \in \Sigma$. Similarly as in the local setting, however without any rescaling, we are able to show the norm resolvent convergence of $H_{\underline{V_\varepsilon}}$ to $H_{V\delta_\Sigma}$ for general interaction matrices under abstract conditions; see Theorem 8.3. For interaction matrices of the form $V = \eta I_N + \tau\beta$, $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, we show in Theorem 8.9 that the simple condition

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < 4, \qquad d = \eta^2 - \tau^2,$$

guarantees the norm resolvent convergence.

Before concluding the introduction, it should be mentioned that in the process of writing this thesis, the language tool DeepL was consulted for stylistic and grammatical improvements concerning the English language.

2 Preliminaries

In this chapter we provide necessary preliminary results. We start by introducing various notations and conventions in Section 2.1. Then, we define in Section 2.2 *special C^2-surfaces* which are roughly speaking subsets of unions of rotated C_b^2-graphs. These surfaces are important because we consider δ-shell potentials and integral operators on such surfaces in the main parts of this thesis; cf. Section 3.1 and Section 3.3. After introducing and constructing Sobolev spaces on these special C^2-surfaces, we study tubular neighbourhoods of such surfaces in Section 2.3. In Section 2.4 we provide elementary definitions and results for Bochner spaces, which turn out to be very useful in the Chapters 4–7, where we consider functions with values on Sobolev spaces on special C^2-surfaces. In Section 2.5 we deal with the norm resolvent convergence of unbounded self-adjoint operators and its spectral implications. Finally, in Section 2.6, we collect conditions for the invertibility of bounded operators, which is crucial for handling resolvent formulas.

2.1 Notations

In this section we provide a list with frequently used notations and conventions throughout this thesis.

(i) By $\theta \in \{2, 3\}$ we denote the space dimension and we set $N = 2$ for $\theta = 2$ and $N = 4$ for $\theta = 3$.

(ii) The symbol Ω_+ denotes an open subset of $\mathbb{R}^\theta$ such that its boundary Σ is a special C^2-surface as in Definition 2.1. In this case we set $\Omega_- := \mathbb{R}^\theta \setminus \overline{\Omega_+}$ and if u is a function defined on $\mathbb{R}^\theta$, then we define $u_\pm := u \upharpoonright \Omega_\pm$.

(iii) For a topological vector space $\mathcal{X}$ the expression $\mathcal{X}'$ denotes its dual space; i.e. the space of all continuous linear functionals defined on $\mathcal{X}$. Moreover, for $x \in \mathcal{X}$ and $x' \in \mathcal{X}'$ we introduce the bilinear duality product

$$ {}_{\mathcal{X}'}\langle x', x \rangle_{\mathcal{X}} := x'(x). $$

If the space $\mathcal{X}$ is equipped with a continuous antilinear conjugation operation $\mathcal{X} \ni x \to \overline{x} \in \mathcal{X}$ fulfilling $\overline{\overline{x}} = x$ for all $x \in \mathcal{X}$, then we introduce the sesquilinear duality product

$$ \langle x', x \rangle_{\mathcal{X}' \times \mathcal{X}} := {}_{\mathcal{X}'}\langle x', \overline{x} \rangle_{\mathcal{X}}. $$

Furthermore, if $\mathcal{Y}$ is also a topological vector space with a continuous antilinear conjugation operation and $\mathcal{A}$ is a linear bounded operator mapping from $\mathcal{X}$ to $\mathcal{Y}$, then the antidual operator $\mathcal{A}' : \mathcal{Y}' \to \mathcal{X}'$ (which is again linear and bounded) is defined by the relationship

$$\langle y', \mathcal{A}x\rangle_{\mathcal{Y}'\times\mathcal{Y}} = \langle \mathcal{A}'y', x\rangle_{\mathcal{X}'\times\mathcal{X}} \quad \forall x \in \mathcal{X}, y' \in \mathcal{Y}'.$$

(iv) Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces and $\mathcal{A}$ be a linear operator mapping from $\mathcal{H}$ to $\mathcal{G}$. The domain, kernel and range of $\mathcal{A}$ are denoted by $\operatorname{dom}\mathcal{A}$, $\ker\mathcal{A}$ and $\operatorname{ran}\mathcal{A}$, respectively. The norm and the scalar product (which is antilinear with respect to the second argument) in $\mathcal{H}$ are expressed by $\|\cdot\|_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{H}}$. If $\mathcal{A}$ is bounded and everywhere defined, then we write $\|\mathcal{A}\|_{\mathcal{H}\to\mathcal{G}}$ for its operator norm. If $\mathcal{H} = \mathcal{G}$ and $\mathcal{A}$ is closed, then the resolvent set, the spectrum and the point spectrum of $\mathcal{A}$ are denoted by $\rho(\mathcal{A})$, $\sigma(\mathcal{A})$ and $\sigma_p(\mathcal{A})$, respectively. Furthermore, if the domain of $\mathcal{A}$ is dense in $\mathcal{H}$, then the adjoint of $\mathcal{A}$ is denoted by $\mathcal{A}^*$ and if $\mathcal{A} = \mathcal{A}^*$, then $\sigma_{\mathrm{ess}}(\mathcal{A})$ and $\sigma_{\mathrm{disc}}(\mathcal{A})$ are the essential and discrete spectrum of $\mathcal{A}$.

(v) Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, J be a countable index set, $\mathcal{A}_j, j \in J$, and $\mathcal{A}$ be bounded linear operators mapping from $\mathcal{H}$ to $\mathcal{G}$. If for every $u \in \mathcal{H}$ and $\delta > 0$ exists a finite index set $J_{\delta,u} \subset J$ such that $\big\|\sum_{j\in J'} \mathcal{A}_j u - \mathcal{A}u\big\|_{\mathcal{H}} < \delta$ for all finite index sets J' with $J' \supset J_{\delta,u}$, then we say that the family $(\mathcal{A}_j)_{j\in J}$ is strongly summable and set $\sum_{j\in J}^{\mathrm{st.}} \mathcal{A}_j := \mathcal{A}$. Furthermore, if $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{G}}$ are also Hilbert spaces and $\mathcal{B} : \mathcal{G} \to \widetilde{\mathcal{G}}$ and $\mathcal{C} : \widetilde{\mathcal{H}} \to \mathcal{H}$ are bounded linear operators, then the family $(\mathcal{B}\mathcal{A}_j\mathcal{C})_{j\in J}$ is strongly summable and $\sum_{j\in J}^{\mathrm{st.}} \mathcal{B}\mathcal{A}_j\mathcal{C} = \mathcal{B}\big(\sum_{j\in J}^{\mathrm{st.}} \mathcal{A}_j\big)\mathcal{C}$.

(vi) The expression $[\cdot,\cdot]$ denotes the commutator of two operators.

(vii) To denote sets of functions, we use symbols having the structure $S(A; B)$, where S reflects the properties of the functions and A and B denote the domain and codomain of the functions, respectively. For example $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, $C^1(\Sigma; \mathbb{R})$, $W^1(\mathbb{R}; \mathbb{C}^{N\times N})$, etc. Moreover, if $S(A)$ is not specified otherwise, then we set $S(A) := S(A; \mathbb{C})$. For example $L^\infty((-1, 1)) := L^\infty((-1, 1); \mathbb{C})$.

(viii) If $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is open, then $H^r(U)$ and $W^r_\infty(U)$ denote the L^2- and L^∞-based Sobolev spaces of order r, respectively; cf. [54, Chaper 3]. Moreover, if Σ is the boundary of a C^2-domain, then $H^r(\Sigma)$ and $W^r_\infty(\Sigma)$ are Sobolev spaces on the boundary Σ; cf. Section 2.2. Vector and matrix-valued Sobolev spaces are defined in the natural way, i.e. component-wise.

(ix) If $k, n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is an open set, then we write $C^k_b(U)$ for the space which contains all $f \in C^k(U)$ such that f and all partial derivatives of f up to order k are bounded. Moreover, we set $C^\infty_b(U) = \bigcap_{k=1}^\infty C^k_b(U)$. If Σ is a C^2-smooth hypersurface and $k \le 2$, then the space $C^k_b(\Sigma)$ is defined via local coordinates. The corresponding spaces of vector and matrix-valued functions are defined component-wise.

(x) If $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ is open, then $C_0^\infty(U)$ denotes the set of all compactly supported C^∞-functions on U. Furthermore, the set $C_0^\infty(\overline{U})$ contains all C^∞-functions on U which have an extension to a function in $C_0^\infty(\mathbb{R}^n)$. Again, the corresponding spaces of vector and matrix-valued functions are defined component-wise.

(xi) The usual $L^2((-1,1))$-based Bochner space of $\mathcal{H}$-valued functions is denoted by $L^2((-1,1);\mathcal{H})$; cf. Section 2.4. For $\mathcal{H} = H^r(S;\mathbb{C}^N)$, where S is either equal to Σ or equal to $\mathbb{R}^{\theta-1}$, we write $\mathcal{B}^r(S)$ instead of $L^2((-1,1);H^r(S;\mathbb{C}^N))$. We also write $\|\cdot\|_r$ for the norm in $\mathcal{B}^r(S)$. In a similar way, we define

$$
\begin{aligned}
\|\cdot\|_{r\to r'} &:= \|\cdot\|_{\mathcal{B}^r(S)\to\mathcal{B}^{r'}(S)},\\
\|\cdot\|_{r\to\mathcal{H}} &:= \|\cdot\|_{\mathcal{B}^r(S)\to\mathcal{H}},\\
\|\cdot\|_{\mathcal{H}\to r'} &:= \|\cdot\|_{\mathcal{H}\to\mathcal{B}^{r'}(S)}.
\end{aligned}
$$

(xii) If $(\mathscr{O},\mathscr{A},\lambda)$ denotes a measure space, and $f:\mathscr{O}\to\mathbb{C}$ is an integrable function, then $\int_{\mathscr{O}} f(t)\,d\lambda(t)$ denotes the integral of f. In the case that λ is the Lebesgue measure we simplify this notation to $\int_{\mathscr{O}} f(t)\,dt$.

(xiii) Following [54, Appendix B], we call $(\mathcal{H}_0,\mathcal{H}_1)$ a compatible pair, if $\mathcal{H}_0$ and $\mathcal{H}_1$ are two Hilbert spaces which are continuously embedded in a bigger Hausdorff topological vector space. In this situation, one can construct with the K-method (or various other methods, see [25, 26, 43, 54], which yield the same spaces with equivalent norms) a family of Hilbert spaces $[\mathcal{H}_0,\mathcal{H}_1]_\tau$, $\tau\in(0,1)$, such that $\mathcal{H}_0\cap\mathcal{H}_1 \subset [\mathcal{H}_0,\mathcal{H}_1]_\tau \subset \mathcal{H}_0+\mathcal{H}_1$ for all $\tau\in(0,1)$. Assume that $(\mathcal{G}_0,\mathcal{G}_1)$ is another compatible pair of Hilbert spaces. Then, for two bounded operators $\mathcal{A}_0:\mathcal{H}_0\to\mathcal{G}_0$ and $\mathcal{A}_1:\mathcal{H}_1\to\mathcal{G}_1$ such that $\mathcal{A}_0u=\mathcal{A}_1u$ for all $u\in\mathcal{H}_0\cap\mathcal{H}_1$, there exists by [54, Theorem B.2] a unique bounded linear operator

$$
\mathcal{A}_\tau : [\mathcal{H}_0,\mathcal{H}_1]_\tau \to [\mathcal{G}_0,\mathcal{G}_1]_\tau
$$

such that $\mathcal{A}_0u=\mathcal{A}_1u=\mathcal{A}_\tau u$ for all $u\in\mathcal{H}_0\cap\mathcal{H}_1$. Moreover, its norm can be estimated by

$$
\|\mathcal{A}_\tau\|_{[\mathcal{H}_0,\mathcal{H}_1]_\tau\to[\mathcal{G}_0,\mathcal{G}_1]_\tau} \le \|\mathcal{A}_0\|^{1-\tau}_{\mathcal{H}_0\to\mathcal{G}_0}\|\mathcal{A}_1\|^{\tau}_{\mathcal{H}_1\to\mathcal{G}_1}.
$$

(xiv) The application of a holomorphic function to a matrix (or a matrix-valued function) A is defined via the associated power series, whenever it converges. This implies for two holomorphic functions f,g that $f(A)g(A)=(fg)(A)$.

(xv) The symbol $|\cdot|$ is used for the absolute value, the Euclidean vector norm or the Frobenius norm of a number, a vector or a matrix, respectively. We write $\langle\cdot,\cdot\rangle$ for the Euclidean scalar product in $\mathbb{C}^n$, $n \in \mathbb{N}$, which is antilinear in the second argument.

(xvi) For $v = (v_1, \dots, v_n)^T \in \mathbb{C}^n$, $n \in \mathbb{N}$, we simply write $v = (v_1, \dots, v_n)$. Similarly, we use the notation $v = (v', v_n)$ for the vector $v = (v'^T, v_n)^T$ with $v' \in \mathbb{C}^{n-1}$ and $v_n \in \mathbb{C}$. Moreover, we set $v[j] = v_j$ for $j \in \{1, \dots, n\}$. Analogously, if $A \in \mathbb{C}^{n\times n}$, then $A[j,k]$, $j,k \in \{1,\dots,n\}$, denotes the (j,k)-th entry of the matrix A.

(xvii) The expression $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^{\theta-1}$. Moreover, $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the partial Fourier transforms in $\mathbb{R}^\theta$ with respect to the first $\theta - 1$ variables and the θ-th variable, respectively. These transforms are given for $\psi \in \mathcal{S}(\mathbb{R}^{\theta-1})$ and $u \in \mathcal{S}(\mathbb{R}^\theta)$ by

$$
\begin{aligned}
\mathcal{F}\psi(\xi') &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} \psi(x') e^{-i\langle x',\xi'\rangle} dx', && \xi' \in \mathbb{R}^{\theta-1},\\
\mathcal{F}_1 u(\xi) &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} u(x',\xi_\theta) e^{-i\langle x',\xi'\rangle} dx', && \xi = (\xi',\xi_\theta) \in \mathbb{R}^\theta,\\
\mathcal{F}_2 u(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi', x_\theta) e^{-ix_\theta\xi_\theta} dx_\theta, && \xi = (\xi',\xi_\theta) \in \mathbb{R}^\theta,
\end{aligned}
$$

and can be uniquely extended to continuous operators in $\mathcal{S}'(\mathbb{R}^{\theta-1})$ and $\mathcal{S}'(\mathbb{R}^\theta)$, where $\mathcal{S}$ denotes the space of tempered distributions; cf. [63, Chapter IX]. Moreover, the application of the Fourier transform to vector and matrix-valued functions or distributions is defined component-wise. The complete Fourier transform in $\mathbb{R}^\theta$ is given by $\mathcal{F}_{1,2} := \mathcal{F}_1\mathcal{F}_2 = \mathcal{F}_2\mathcal{F}_1$.

(xviii) The letter $C > 0$ always denotes a generic constant which may change in-between lines.

(xix) The branch of the square root is fixed by $\operatorname{Im}\sqrt{w} > 0$ for $w \in \mathbb{C} \setminus [0,\infty)$.

(xx) The *tangens cardinalis* $\operatorname{tanc} : \mathbb{C} \setminus \{n\pi + \frac{\pi}{2} : n \in \mathbb{Z}\} \to \mathbb{C}$ is defined by

$$
\operatorname{tanc}(w) := \begin{cases} \frac{\tan(w)}{w}, & w \in \mathbb{C} \setminus \big(\{0\} \cup \{n\pi + \frac{\pi}{2} : n \in \mathbb{Z}\}\big),\\ 1, & w = 0. \end{cases}
$$

For $x \in \mathbb{R} \setminus \{0\}$ the equation $\operatorname{tanc}(ix) = \frac{\tanh(x)}{x}$ is valid.

(xxi) The expression $\mathrm{SO}(\theta)$ denotes the rotation matrices in $\mathbb{R}^{\theta\times\theta}$, i.e. the real $\theta \times \theta$ orthogonal matrices which have determinant one.

2.2 Special C^2-surfaces and corresponding Sobolev spaces

We introduce and study in this section, which is based on [14, Section 2.1], so-called *special C^2-surfaces* in $\mathbb{R}^\theta$. Moreover, by a partition of unity and suitable parametrizations we are able to define Sobolev spaces on such surfaces.

Definition 2.1. *Let $\Omega_+ \subset \mathbb{R}^\theta$, $\theta \in \{2,3\}$, be an open set and $\Sigma := \partial\Omega_+$. Then, we call Σ a special C^2-surface if there exist open sets $W_1, \dots, W_p \subset \mathbb{R}^\theta$, mappings $\zeta_1, \dots, \zeta_p \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$, rotation matrices $\kappa_1, \dots, \kappa_p \in \mathrm{SO}(\theta)$, and an $\varepsilon_\Sigma > 0$ such that the following is true:*

(i) $\Sigma \subset \bigcup_{l=1}^p W_l$.

(ii) *If $x \in \partial\Omega_+ = \Sigma$, then there exists $l \in \{1, \dots, p\}$ such that $B(x, \varepsilon_\Sigma) \subset W_l$.*

(iii) *$W_l \cap \Omega_+ = W_l \cap \Omega_l$, where $\Omega_l = \{\kappa_l\big(x', x_\theta\big) : x_\theta < \zeta_l(x'),\, (x', x_\theta) \in \mathbb{R}^\theta\}$, for $l \in \{1, \dots, p\}$.*

Furthermore, in this case we define the sets $\Sigma_l := \partial\Omega_l = \{\kappa_l(x', \zeta_l(x')) : x' \in \mathbb{R}^{\theta-1}\}$ and $\Omega_- := \mathbb{R}^\theta \setminus \overline{\Omega_+}$, denote the unit normal vector field at Σ that is pointing outwards of Ω_+ by ν and use the expression σ for the $(\theta-1)$-dimensional Hausdorff measure restricted to Σ.

One can check easily that compact C^2-hypersurfaces and C_b^2-graphs are special C^2-surfaces. This class is essentially the intersection of the hypersurfaces described by [70, Chapter VI, Section 3.3] and [7, Definition 2.1 and Hypothesis 2.3]. The assumptions in [70] guarantee us on the one hand the existence of suitable trace and extension theorems, see Proposition 2.3 and Proposition 2.6, and on the other hand the assumptions in [7] imply that the ε-tubular neighbourhood can be identified with the set $\Sigma \times (-\varepsilon, \varepsilon)$; see Proposition 2.12. Both are essential in the later parts of this thesis.

Next, we introduce Sobolev spaces on special C^2-surfaces. We recall that Sobolev spaces on open sets are defined as in [54, Chapter 3]. To transfer the definitions to Σ, we choose the partition of unity $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta)$ for Σ subordinate to $W_1, \dots, W_p$ from Proposition A.4. We define for $l \in \{1, \dots, p\}$

$$\varkappa_l : \mathbb{R}^{\theta-1} \to \Sigma_l, \qquad \varkappa_l(x') := \kappa_l(x', \zeta_l(x')), \tag{2.1}$$

and for $\psi \in L^2(\Sigma; \mathbb{C}^N)$ we write

$$\psi_{\Sigma_l}(x') := \begin{cases} \varphi_l(\varkappa_l(x'))\psi(\varkappa_l(x')), & \varkappa_l(x') \in \Sigma, \\ 0, & \varkappa_l(x') \notin \Sigma. \end{cases}$$

Then, $\psi_{\Sigma_l} \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and $\psi(x_\Sigma) = \sum_{l=1,x_\Sigma\in\Sigma_l}^{p} \psi_{\Sigma_l}(\varkappa_l^{-1}(x_\Sigma))$ for $x_\Sigma \in \Sigma$. As usual, we introduce for $r \in [0,2]$

$$H^r(\Sigma;\mathbb{C}^N) := \big\{\psi \in L^2(\Sigma;\mathbb{C}^N) : \psi_{\Sigma_l} \in H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N) \text{ for all } l = 1,\dots,p\big\} \tag{2.2}$$

and endow this space with the scalar product

$$\langle \phi,\psi\rangle_{H^r(\Sigma;\mathbb{C}^N)} = \sum_{l=1}^{p} \langle \phi_{\Sigma_l},\psi_{\Sigma_l}\rangle_{H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N)}, \qquad \phi,\psi \in H^r(\Sigma;\mathbb{C}^N).$$

Sobolev spaces $H^r(\Sigma;\mathbb{C}^N)$ with $r \in [-2,0)$ are defined by duality. Furthermore, setting $U_l := \varkappa_l^{-1}(\Sigma \cap W_l)$ allows us to define for $\mathcal{V} \in \{\mathbb{C};\mathbb{C}^{N\times N}\}$ and $k \in \{1,2\}$ the Sobolev space

$$W^k_\infty(\Sigma;\mathcal{V}) := \big\{F \in L^\infty(\Sigma;\mathcal{V}) : (F\circ\varkappa_l)\restriction U_l \in W^k_\infty(U_l;\mathcal{V}) \text{ for all } l = 1,\dots,p\big\}$$

and equip it with the norm

$$\|F\|_{W^k_\infty(\Sigma;\mathcal{V})} := \max_{l\in\{1,\dots,p\}} \|(F\circ\varkappa_l)\restriction U_l\|_{W^k_\infty(U_l;\mathcal{V})}, \quad F \in W^k_\infty(\Sigma;\mathcal{V}).$$

Since the Sobolev spaces on Σ are defined via Sobolev spaces on open sets, one can check that $H^r(\Sigma;\mathbb{C}^N)$ is a Hilbert space and $W^k_\infty(\Sigma;\mathcal{V})$ is a Banach space. We state useful properties of the just introduced Sobolev spaces on Σ in the next proposition.

Proposition 2.2. *Let $\mathcal{V} \in \{\mathbb{C},\mathbb{C}^{N\times N}\}$, $\Sigma \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1, $k \in \{1,2\}$ and $r, r_1, r_2 \in [-2,2]$. Then, the following statements hold:*

(i) *If $r = (1-\tau)r_1 + \tau r_2$, then $H^r(\Sigma;\mathbb{C}^N) = [H^{r_1}(\Sigma;\mathbb{C}^N), H^{r_2}(\Sigma;\mathbb{C}^N)]_\tau$.*

(ii) *If $0 \le r_1 \le r_2$, then $H^{r_2}(\Sigma;\mathbb{C}^N)$ is densely contained in $H^{r_1}(\Sigma;\mathbb{C}^N)$.*

(iii) *If $|r| \le k$, $\psi \in H^r(\Sigma;\mathbb{C}^N)$ and $F \in W^k_\infty(\Sigma;\mathcal{V})$, then $F\psi \in H^r(\Sigma;\mathbb{C}^N)$ and*

$$\|F\psi\|_{H^r(\Sigma;\mathbb{C}^N)} \le C\|F\|_{W^k_\infty(\Sigma;\mathcal{V})}\|\psi\|_{H^r(\Sigma;\mathbb{C}^N)},$$

where $C > 0$ does not depend on ψ and F.

(iv) *If $F, G \in W^k_\infty(\Sigma;\mathcal{V})$, then $FG \in W^k_\infty(\Sigma;\mathcal{V})$ and*

$$\|FG\|_{W^k_\infty(\Sigma;\mathcal{V})} \le C\|F\|_{W^k_\infty(\Sigma;\mathcal{V})}\ \|G\|_{W^k_\infty(\Sigma;\mathcal{V})},$$

where $C > 0$ does not depend on F and G.

(v) *If $I \subset \mathbb{R}$ is an open interval, $F \in C^1_b(I;\mathcal{V})$, $G \in W^1_\infty(\Sigma)$ and $G(\Sigma) \subset I$, then $F\circ G \in W^1_\infty(\Sigma;\mathcal{V})$ and*

$$\|F\circ G\|_{W^1_\infty(\Sigma;\mathcal{V})} \le\ C\|F\|_{W^1_\infty(I;\mathcal{V})}\|G\|_{W^1_\infty(\Sigma)},$$

where $C > 0$ does not depend on F and G.

Proof. We start by showing (i). This can be found in [54, Theorem B.11] for the case that Σ is compact. The proof in [54] works as follows. First, the case of graphs is considered and then a partition of unity $\varphi_1, \dots, \varphi_p \in C_0^\infty(\mathbb{R}^\theta)$ for Σ subordinate to $W_1, \dots, W_p$ and functions $\chi_1, \dots, \chi_p \in C_0^\infty(\mathbb{R}^\theta)$ which fulfil $\operatorname{supp} \chi_l \subset W_l$ and $\phi_l \chi_l = \varphi_l$ for $l \in \{1, \dots, p\}$ are used to transfer the results to compact hypersurfaces. However, the proof does not change if the functions $\varphi_1, \dots, \varphi_p$ and $\chi_1, \dots, \chi_p$ are assumed to be in $C_b^\infty(\mathbb{R}^\theta)$ instead of $C_0^\infty(\mathbb{R}^\theta)$ and hence Proposition A.4 shows that (i) is also valid if Σ is a special C^2-surface.

Next, let us consider (ii). By [54, the lines above eq. (3.22)] $H^{r_1}(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ is densely contained in $H^{r_2}(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$. Using this knowledge and the definition of $H^r(\Sigma; \mathbb{C}^N)$ in (2.2) via the partition of unity $\varphi_1, \dots \varphi_p$ from Proposition A.4 one concludes that also $H^{r_2}(\Sigma; \mathbb{C}^N)$ is densely contained in $H^{r_1}(\Sigma; \mathbb{C}^N)$.

To prove (iii), we first assume that r is a nonnegative integer, then by applying the product rule for weak derivatives, see [29, Section 4.2.2], we get

$$
\begin{aligned}
\|F\psi\|^2_{H^r(\Sigma;\mathbb{C}^N)} &= \sum_{l=1}^p \|(F \circ \varkappa_l)\psi_{\Sigma_l}\|^2_{H^r(U_l;\mathbb{C}^N)} \\
&\leq C \sum_{l=1}^p \|F \circ \varkappa_l\|^2_{W^k_\infty(U_l;\mathcal{V})} \|\psi_{\Sigma_l}\|^2_{H^r(U_l;\mathbb{C}^N)} \\
&\leq C \|F\|^2_{W^k_\infty(\Sigma;\mathcal{V})} \sum_{l=1}^p \|\psi_{\Sigma_l}\|^2_{H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\
&= C \|F\|^2_{W^k_\infty(\Sigma;\mathcal{V})} \|\psi\|^2_{H^r(\Sigma;\mathbb{C}^N)}.
\end{aligned}
$$

If $r \in [0, k]$, then the result follows by interpolation from (i) and Section 2.1 (xiii). If $r \in [-k, 0)$, then $F\psi$ is defined by (anti-)duality via

$$
\langle F\psi, \gamma \rangle_{H^r(\Sigma;\mathbb{C}^N) \times H^{-r}(\Sigma;\mathbb{C}^N)} := \langle \psi, F^*\gamma \rangle_{H^r(\Sigma;\mathbb{C}^N) \times H^{-r}(\Sigma;\mathbb{C}^N)} \qquad \forall \gamma \in H^{-r}(\Sigma; \mathbb{C}^N).
$$

Thus, the result is a consequence of the case where r is positive.

Items (iv) and (v) follow from the definition of $W^k_\infty(\Sigma; \mathcal{V})$ via Sobolev spaces on open sets and the product rule and chain rule for weak derivatives; see again [29, Section 4.2.2]. □

Next, we formulate a suitable trace theorem for special C^2-surfaces.

Proposition 2.3. *Let $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$, $\theta \in \{2, 3\}$, be a special C^2-surface as in Definition 2.1 and $r \in (\frac{1}{2}, \frac{5}{2})$. Then, the following is true:*

(i) *There exists a unique bounded and surjective operator*

$$\boldsymbol{t}_\Sigma^\pm : H^r(\Omega_\pm;\mathbb{C}^N) \to H^{r-1/2}(\Sigma;\mathbb{C}^N)$$

such that $\boldsymbol{t}_\Sigma^\pm u = u \upharpoonright \Sigma$ *for all* $u \in H^r(\Omega_\pm;\mathbb{C}^N) \cap C(\overline{\Omega_\pm};\mathbb{C}^N)$.

(ii) *There exists a unique bounded and surjective operator*

$$\boldsymbol{t}_\Sigma : H^r(\mathbb{R}^\theta;\mathbb{C}^N) \to H^{r-1/2}(\Sigma;\mathbb{C}^N)$$

such that $\boldsymbol{t}_\Sigma u = u \upharpoonright \Sigma$ *for all* $u \in H^r(\mathbb{R}^\theta;\mathbb{C}^N) \cap C(\mathbb{R}^\theta;\mathbb{C}^N)$.

Proof. Item (i) follows from [21, Theorem 8.7] and [48, Theorem 2]. Moreover, defining $\boldsymbol{t}_\Sigma u := \boldsymbol{t}_\Sigma^+ u_+$ for $u \in H^1(\mathbb{R};\mathbb{C}^N)$ shows that there exists a bounded surjective operator satisfying $\boldsymbol{t}_\Sigma u = u \upharpoonright \Sigma$ for $u \in H^r(\mathbb{R}^\theta;\mathbb{C}^N) \cap C(\mathbb{R}^\theta;\mathbb{C}^N)$. The operator $\boldsymbol{t}_\Sigma$ is unique since the set $H^r(\mathbb{R}^\theta;\mathbb{C}^N) \cap C(\mathbb{R}^\theta;\mathbb{C}^N)$ is dense $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$; see for instance [54, the text above eq. (3.22)]. Thus, also (ii) is true. □

Remark 2.4. *For* $u \in H^r(\mathbb{R}^\theta \setminus \Sigma;\mathbb{C}^N) = H^r(\Omega_+;\mathbb{C}^N) \oplus H^r(\Omega_-;\mathbb{C}^N)$ *we define* $\boldsymbol{t}_\Sigma^\pm u := \boldsymbol{t}_\Sigma^\pm u_\pm$, *where* $u_\pm := u \upharpoonright \Omega\pm$.

Corollary 2.5. *Let* $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ *be a special* C^2*-surface as in Definition 2.1,* ν *be the unit normal vector field pointing outwards of* Ω_+, $f, g \in H^1(\Omega_\pm;\mathbb{C}^N)$ *and* $j \in \{1, \dots, \theta\}$. *Then,*

$$(\partial_j f, g)_{L^2(\Omega_\pm;\mathbb{C}^N)} + (f, \partial_j g)_{L^2(\Omega_\pm;\mathbb{C}^N)} = \pm(\nu[j]\boldsymbol{t}_\Sigma^\pm f, \boldsymbol{t}_\Sigma^\pm g)_{L^2(\Sigma;\mathbb{C}^N)},$$

where $\nu[j]$ *denotes the* j*-th component of* ν.

Proof. If $f, g \in C_0^\infty(\overline{\Omega_\pm};\mathbb{C}^N)$, then the statement is a consequence of the divergence theorem. The divergence theorem for Lipschitz domains with compact boundaries is given by [54, Theorem 3.34]. However, as $f, g \in C_0^\infty(\overline{\Omega_\pm};\mathbb{C}^N)$, i.e. they have compactly supported C^∞-extensions to $\mathbb{R}^\theta$, the divergence theorem for compact boundaries is also applicable in our setting. According to [54, the text below (3.23)] $C_0^\infty(\overline{\Omega_\pm};\mathbb{C}^N)$ is dense in $H^1(\Omega_\pm;\mathbb{C}^N)$. Thus, the result follows from the continuity of the trace operator. □

Under our geometric assumptions we not only have a suitable trace theorem, additionally, according to the upcoming proposition there also exists an extension operator extending functions from $H^r(\Omega_\pm;\mathbb{C}^N)$ to functions in $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$.

Proposition 2.6. *Let* $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ *be a special* C^2*-surface as in Definition 2.1. Then, there exists a bounded extension operator (called Stein's extension operator)* $E^\pm : L^2(\Omega_\pm;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ *which satisfies* $(Eu_\pm) \upharpoonright \Omega_\pm = u_\pm$ *for all functions* $u \in L^2(\Omega_\pm;\mathbb{C}^N)$ *and also acts as a bounded operator from* $H^r(\Omega_\pm;\mathbb{C}^N)$ *to* $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$ *for all* $r \geq 0$.

Proof. If $r \in \mathbb{N}_0$, then the proof follows from [70, Chapter VI, Section 3.1, Theorem 5] since $\Omega_\pm$ satisfies the conditions in [70, Chapter VI, Section 3.3]. If $r \notin \mathbb{N}_0$, then interpolation, see [54, Theorem B.7 and Theorem B.8] and Section 2.1 (xiii), yields the claim. □

2.3 Tubular neighbourhoods of special C^2-surfaces

In this section, which contains parts of [14, Section 2.1 and Appendix A], we study tubular neighbourhoods of Σ. This is important, as we define in (4.3) so-called strongly localized potentials on such sets.

Definition 2.7. *Let $\Sigma \subset \mathbb{R}^\theta$ be a special C^2-surface, ν, Σ_l and Ω_l be as in Definition 2.1, and $\varkappa_l$ be defined by* (2.1)*. Then, we set*

$$\iota : \Sigma \times \mathbb{R} \to \mathbb{R}^\theta, \qquad \iota(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma),$$

and for $\varepsilon > 0$ we call $\Omega_\varepsilon := \iota(\Sigma \times (-\varepsilon, \varepsilon))$ the (ε-)tubular neighbourhood of Σ. Moreover, we define for $l \in \{1, \dots, p\}$ the function

$$\iota_l : \mathbb{R}^{\theta-1} \times \mathbb{R} \to \mathbb{R}^\theta, \qquad \iota_l(x', t) := \varkappa_l(x') + t\nu_l(\varkappa_l(x')),$$

where ν_l denotes the unit normal vector field on Σ_l pointing outwards of Ω_l.

Note that $\iota(\varkappa_l(x'), t) = \iota_l(x', t)$ for all $x' \in \varkappa_l^{-1}(\Sigma)$, $t \in \mathbb{R}$ and $l \in \{1, \dots, p\}$, and

$$\nu_l(\varkappa_l(x')) = \frac{\kappa_l(-\nabla\zeta_l(x'), 1)}{\sqrt{1 + |\nabla\zeta_l(x')|^2}} \qquad \forall x' \in \mathbb{R}^{\theta-1},$$

with ζ_l from Definition 2.1. From now on we write $\nu_l(x')$ instead of $\nu_l(\varkappa_l(x'))$ for $x' \in \mathbb{R}^{\theta-1}$ in order to simplify notation.

Before we study the maps ι and ι_l in detail, we provide a useful variant of the mean value theorem for vector and matrix-valued functions.

Lemma 2.8. *Let $k, l, n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ be an open set and $A \in C_b^1(U; \mathbb{C}^{k\times l})$. If $x, y \in U$ and the line segment connecting x and y is contained in U, then*

$$\begin{aligned} |A(x) - A(y)| &\leq \sup_{\mu\in[0,1]} \Big(\sum_{j=1}^n |(\partial_j A)(x + \mu(y-x))|^2 \Big)^{1/2} |x - y| \\ &\leq \sqrt{n} \sup_{\mu\in[0,1], j\in\{1,\dots,n\}} |(\partial_j A)(x + \mu(y-x))||x-y|. \end{aligned}$$

In particular, if $U = \mathbb{R}^n$ and $l = 1$, then

$$|A(x) - A(y)| \leq \|DA\|_{L^\infty(\mathbb{R}^n; \mathbb{C}^{k\times n})} |x - y| \quad \forall x, y \in \mathbb{R}^n.$$

Proof. Recall that $|\cdot|$ denotes, depending on the argument, the absolute value, the Euclidean vector norm, or the Frobenius matrix norm. The fundamental theorem of calculus and the Cauchy-Schwarz inequality lead to

$$\begin{aligned}|A(x)-A(y)| &= \left|\int_0^1 \sum_{j=1}^n (\partial_j A)(x+\mu(y-x))\,(x-y)[j]\,d\mu\right| \\ &\le \int_0^1 \sum_{j=1}^n |(\partial_j A)(x+\mu(y-x))||(x-y)[j]|\,d\mu \\ &\le \sup_{\mu\in[0,1]} \sum_{j=1}^n |(\partial_j A)(x+\mu(y-x))||(x-y)[j]| \\ &\le \sup_{\mu\in[0,1]} \Big(\sum_{j=1}^n |(\partial_j A)(x+\mu(y-x))|^2\Big)^{1/2} |x-y| \\ &\le \sqrt{n} \sup_{\mu\in[0,1], j\in\{1,\dots,n\}} |(\partial_j A)(x+\mu(y-x))||x-y|.\end{aligned}$$

The estimate for the special case $U=\mathbb{R}^n$ and $l=1$ is an immediate consequence of the above estimate. $\square$

Proposition 2.9. *Let $\Sigma\subset\mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1, and let ι and ι_l, $l\in\{1,\dots,p\}$, be as in Definition 2.7. Then, there exists an $\varepsilon_\iota>0$ and constants $C_{\iota,1}, C_{\iota,2}>0$ such that the following holds:*

(i) *For all $x',y'\in\mathbb{R}^{\theta-1}$, $t,s\in(-\varepsilon_\iota,\varepsilon_\iota)$ and $l\in\{1,\dots,p\}$ we have*

$$C_{\iota,1}^{-1}\big(|x'-y'|+|t-s|\big) \le |\iota_l(x',t)-\iota_l(y',s)| \le C_{\iota,1}\big(|x'-y'|+|t-s|\big).$$

(ii) *For all $x_\Sigma, y_\Sigma\in\Sigma$ and $t,s\in(-\varepsilon_\iota,\varepsilon_\iota)$ we have*

$$C_{\iota,2}^{-1}\big(|x_\Sigma-y_\Sigma|+|t-s|\big) \le |\iota(x_\Sigma,t)-\iota(y_\Sigma,s)| \le C_{\iota,2}\big(|x_\Sigma-y_\Sigma|+|t-s|\big).$$

Proof. (i) Let $x',y'\in\mathbb{R}^{\theta-1}$ and $t,s\in(-\varepsilon_\iota,\varepsilon_\iota)$ be fixed, where $\varepsilon_\iota>0$ is, at the moment, a fixed number. Using Definition 2.7, Lemma 2.8, $\varkappa_l(x')=\kappa_l(x',\zeta_l(x'))$ and $\varkappa_l(y')=\kappa_l(y',\zeta_l(y'))$ we find

$$\begin{aligned}&|\iota_l(x',t)-\iota_l(y',s)| \le |\varkappa_l(x')-\varkappa_l(y')|+|t\nu_l(x')-s\nu_l(y')| \\ &\quad\le |\varkappa_l(x')-\varkappa_l(y')|+|t||\nu_l(x')-\nu_l(y')|+|t-s| \\ &\quad\le |x'-y'|+|\zeta_l(x')-\zeta_l(y')|+\varepsilon_\iota|\nu_l(x')-\nu_l(y')|+|t-s| \\ &\quad\le |x'-y'|+\|\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})}|x'-y'| \\ &\qquad\qquad + \varepsilon_\iota\|D\nu_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta\times(\theta-1)})}|x'-y'|+|t-s| \\ &\quad\le \big(1+\|\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})}+\varepsilon_\iota\|D\nu_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta\times(\theta-1)})}\big)\big(|x'-y'|+|t-s|\big).\end{aligned}$$

Now, the second estimate in (i) follows if we fix $0 < \varepsilon_\iota \le 1$ and choose

$$C_{\iota,1} \ge 1 + \max_{l\in\{1,\dots,p\}} \big(\|\nabla \zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})} + \|D\nu_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta\times(\theta-1)})} \big),$$

which is finite since we assumed in Definition 2.1 that $\zeta_l \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$, $l \in \{1,\dots,p\}$.

Next, we prove the first inequality in (i). We start by rewriting

$$\begin{aligned} |\iota_l(x',t) - \iota_l(y',s)|^2 &= |\varkappa_l(x') - \varkappa_l(y')|^2 \\ &\quad + 2\langle \varkappa_l(x') - \varkappa_l(y'), t\nu_l(x') - s\nu_l(y')\rangle + |t\nu_l(x') - s\nu_l(y')|^2. \end{aligned} \tag{2.3}$$

We estimate all three terms on the right-hand side separately. For the first one, we find with $\varkappa_l(x') = \kappa_l(x',\zeta_l(x'))$, $\varkappa_l(y') = \kappa_l(y',\zeta_l(y'))$, and as $\kappa_l \in \mathrm{SO}(\theta)$ that

$$|\varkappa_l(x') - \varkappa_l(y')|^2 = |x'-y'|^2 + |\zeta_l(x') - \zeta_l(y')|^2 \ge |x'-y'|^2. \tag{2.4}$$

Next, we consider the second term on the right-hand side of (2.3). We start by observing

$$\begin{aligned} \mathcal{I} &:= \big|\langle \varkappa_l(x') - \varkappa_l(y'), t\nu_l(x')\rangle\big| \\ &= \left| \frac{1}{\sqrt{1+|\nabla\zeta_l(x')|^2}} \left\langle \kappa_l \begin{pmatrix} x'-y' \\ \zeta_l(x') - \zeta_l(y') \end{pmatrix}, t\kappa_l \begin{pmatrix} -\nabla\zeta_l(x') \\ 1 \end{pmatrix} \right\rangle \right| \\ &= \left| t \frac{\langle x'-y', -\nabla\zeta_l(x')\rangle + \zeta_l(x') - \zeta_l(y')}{\sqrt{1+|\nabla\zeta_l(x')|^2}} \right|. \end{aligned}$$

The mean value theorem shows $\zeta_l(x') - \zeta_l(y') = \langle x'-y', \nabla\zeta_l(x' + \mu(y'-x'))\rangle$ for some $\mu \in [0,1]$. Using Lemma 2.8 the above expression can be further estimated by

$$\begin{aligned} \mathcal{I} &\le \sup_{\mu\in[0,1]} \left| t \frac{\langle x'-y', \nabla\zeta_l(x') - \nabla\zeta_l(x' + \mu(y'-x'))\rangle}{\sqrt{1+|\nabla\zeta_l(x')|^2}} \right| \\ &\le \sup_{\mu\in[0,1]} \varepsilon_\iota |x'-y'||\mu(y'-x')|\|D\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{(\theta-1)\times(\theta-1)})} \\ &\le \varepsilon_\iota |x'-y'|^2 \|D\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{(\theta-1)\times(\theta-1)})}. \end{aligned}$$

Similarly, one has

$$|\langle \varkappa_l(x') - \varkappa_l(y'), s\nu_l(y')\rangle| \le \varepsilon_\iota |x'-y'|^2 \|D\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{(\theta-1)\times(\theta-1)})},$$

and thus

$$\begin{aligned} &2\langle \varkappa_l(x') - \varkappa_l(y'), t\nu_l(x') - s\nu_l(y')\rangle \\ &\qquad \ge -4\varepsilon_\iota |x'-y'|^2 \|D\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{(\theta-1)\times(\theta-1)})}. \end{aligned} \tag{2.5}$$

To estimate the third term on the right-hand side in (2.3), we use Lemma 2.8 as well as $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ for $a, b > 0$ and calculate

$$\begin{aligned} |t\nu_l(x') - s\nu_l(y')|^2 &= |(t-s)\nu_l(x') - s(\nu_l(y') - \nu_l(x'))|^2 \\ &\geq \big(|t-s| - |s(\nu_l(y') - \nu_l(x'))|\big)^2 \\ &\geq \frac{1}{2}|t-s|^2 - s^2|\nu_l(y') - \nu_l(x')|^2 \\ &\geq \frac{1}{2}|t-s|^2 - \varepsilon_\iota^2|x'-y'|^2\|D\nu_l\|^2_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta\times(\theta-1)})}. \end{aligned} \tag{2.6}$$

By plugging (2.4)–(2.6) into (2.3) we obtain

$$\begin{aligned} |\iota_l(x',t) - \iota_l(y',s)|^2 &\geq \frac{1}{2}|t-s|^2 \\ &+ |x'-y'|^2\Big(1 - 4\varepsilon_\iota\|D\nabla\zeta_l\|_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{(\theta-1)\times(\theta-1)})} - \varepsilon_\iota^2\|D\nu_l\|^2_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta\times(\theta-1)})}\Big). \end{aligned}$$

As before we conclude from $\zeta_l \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ that for $\varepsilon_\iota > 0$ sufficiently small and $C_{\iota,1} > 0$ sufficiently large the first inequality in (i) is also fulfilled.

(ii) We fix $x_\Sigma, y_\Sigma \in \Sigma$ and $t, s \in (-\varepsilon_\iota, \varepsilon_\iota)$. Let us first assume that $x_\Sigma, y_\Sigma \in \Sigma_l$ for some $l \in \{1, \ldots, p\}$. Then, there exist $x', y' \in \mathbb{R}^{\theta-1}$ such that $x_\Sigma = \varkappa_l(x')$ and $y_\Sigma = \varkappa_l(y')$, and therefore $\iota(x_\Sigma, t) = \iota_l(x', t)$ and $\iota(y_\Sigma, s) = \iota_l(y', s)$. In this case we see

$$|x_\Sigma - y_\Sigma| = \sqrt{|x'-y'|^2 + |\zeta_l(x') - \zeta_l(y')|^2}$$

and therefore combining

$$|x'-y'| \leq |x_\Sigma - y_\Sigma| \leq |x'-y'|\sqrt{1 + \|\nabla\zeta_l\|^2_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})}}$$

with (i) yields (ii). It remains to consider the case where $x_\Sigma, y_\Sigma \in \Sigma$ and there is no $l \in \{1, \ldots, p\}$ such that $x_\Sigma, y_\Sigma \in \Sigma_l$. Then, (ii) and (iii) from Definition 2.1 imply $|x_\Sigma - y_\Sigma| \geq \varepsilon_\Sigma$, where ε_Σ is the number specified in Definition 2.1. We choose $\varepsilon_\iota \leq \varepsilon_\Sigma/6$. Then, $|x_\Sigma - y_\Sigma| \geq 6\varepsilon_\iota$, $|t\nu(x_\Sigma) - s\nu(y_\Sigma)| \leq 2\varepsilon_\iota$ and $|t-s| \leq 2\varepsilon_\iota$ yield

$$|\iota(x_\Sigma, t) - \iota(y_\Sigma, s)| \leq |x_\Sigma - y_\Sigma| + 2\varepsilon_\iota \leq \frac{4}{3}|x_\Sigma - y_\Sigma| \leq \frac{4}{3}(|x_\Sigma - y_\Sigma| + |t-s|)$$

and

$$\begin{aligned} \frac{1}{2}\big(|x_\Sigma - y_\Sigma| + |t-s|\big) &\leq \frac{|x_\Sigma - y_\Sigma|}{2} + \varepsilon_\iota = \frac{|x_\Sigma - y_\Sigma|}{2} + 3\varepsilon_\iota - 2\varepsilon_\iota \\ &\leq |x_\Sigma - y_\Sigma| - 2\varepsilon_\iota \leq |\iota(x_\Sigma, t) - \iota(y_\Sigma, s)|, \end{aligned}$$

which imply (ii) also in this case. □

Eventually, we state a useful consequence of Proposition 2.9.

Corollary 2.10. *Let $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1 and let ε_ι be as in Proposition 2.9. Then, the following holds:*

(i) *For any $x_\Sigma \in \Sigma$ and $t \in (0, \varepsilon_\iota)$ one has $x_\Sigma + t\nu(x_\Sigma) \in \Omega_-$.*

(ii) *For any $x_\Sigma \in \Sigma$ and $t \in (-\varepsilon_\iota, 0)$ one has $x_\Sigma + t\nu(x_\Sigma) \in \Omega_+$.*

Proof. We only show item (i), the proof of assertion (ii) follows along the same lines. We verify the claim by an indirect proof. Assume that there are $x_\Sigma \in \Sigma$ and $t \in (0, \varepsilon_\iota)$ such that $x_\Sigma + t\nu(x_\Sigma) \notin \Omega_-$. Since ν is pointing outwards of Ω_+, we have for small $\mu > 0$ that $x_\Sigma + \mu t\nu(x_\Sigma) \in \Omega_-$. By continuity, this implies that there exists $\mu_0 \in (0, 1]$ such that $x_\Sigma + \mu_0 t\nu(x_\Sigma) \in \Sigma$. However, we obtain from Proposition 2.9 for all $y_\Sigma \in \Sigma$ with a constant $C_{\iota,2} > 0$ the inequality

$$|x_\Sigma + \mu_0 t\nu(x_\Sigma) - y_\Sigma| \geq C_{\iota,2}^{-1}\mu_0 t > 0;$$

this is a contradiction. □

Proposition 2.9 shows that ι is a bi-Lipschitz mapping on $\Sigma \times (-\varepsilon_\iota, \varepsilon_\iota)$. In particular, ι is injective on $\Sigma \times (-\varepsilon_\iota, \varepsilon_\iota)$ and thus $\iota \upharpoonright \Sigma \times (-\varepsilon_\iota, \varepsilon_\iota)$ is a bijection between $\Omega_{\varepsilon_\iota}$ and $\Sigma \times (-\varepsilon_\iota, \varepsilon_\iota)$. In Proposition 2.12 we show that we can also identify Lebesgue spaces on these sets with one another. Before we do so, let us introduce the Weingarten map (or shape operator) on Σ.

Definition 2.11. *Let $\Sigma \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1 and denote for $x_\Sigma = \varkappa_l(x') \in \Sigma$, $l \in \{1, \dots, p\}$, the tangent hyperplane of Σ in the point x_Σ by the symbol $T_{x_\Sigma} = \operatorname{span}\{\partial_j \varkappa_l(x') : j = 1, \dots, \theta - 1\}$. The Weingarten map is the linear operator $W(x_\Sigma) : T_{x_\Sigma} \to T_{x_\Sigma}$ defined by*

$$W(x_\Sigma)\partial_j \varkappa_l(x') = -\partial_j \nu(\varkappa_l(x')), \qquad j \in \{1, \dots, \theta - 1\}.$$

Using the chain rule and $|\nu(x_\Sigma)| = 1$ it is easy to show that $W(x_\Sigma)$ is well-defined, i.e. it is independent of the parametrization $\varkappa_l$ and $-\partial_j \nu(\varkappa_l(x')) \in T_{x_\Sigma}$; see also [47, Lemma 3.9].

Furthermore, we denote the matrix representation of $W(x_\Sigma)$ corresponding to the basis $\{\partial_j \varkappa_l(x') : j = 1, \dots, \theta - 1\}$ of T_{x_Σ} by $L_l(x')$. Then, the eigenvalues of $W(x_\Sigma)$ and $L_l(x')$ coincide and therefore the expression

$$\det(I - tW(x_\Sigma)) := \det(I_{\theta-1} - tL_l(x')), \qquad t \in \mathbb{R},$$

is well-defined. In the next proposition we state important properties of ι and W, and identify $L^1(\Omega_\varepsilon)$ with $L^1(\Sigma \times (-\varepsilon, \varepsilon))$.

Proposition 2.12. *Let $\Sigma \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1 and let ι be as in Definition 2.7. Then, there exists an $\varepsilon_{\mathrm{tub}} \in (0, \varepsilon_\iota)$ such that the following is true:*

(i) $\iota \restriction \Sigma \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$ *is injective.*

(ii) *There exists a $C > 0$ such that $|1 - \det(I - \varepsilon W(x_\Sigma))| \leq C\varepsilon < 1/2$ for all $x_\Sigma \in \Sigma$ and $\varepsilon \in (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$.*

(iii) *For $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$ one has $u \circ (\iota \restriction \Sigma \times (-\varepsilon, \varepsilon)) \in L^1(\Sigma \times (-\varepsilon, \varepsilon))$ if and only if $u \in L^1(\Omega_\varepsilon)$ and in this case*

$$\int_{\Omega_\varepsilon} u(y)\, dy = \int_{-\varepsilon}^{\varepsilon} \int_\Sigma u(y_\Sigma + s\nu(y_\Sigma)) \det(I - sW(y_\Sigma))\, d\sigma(y_\Sigma)\, ds.$$

Proof. Let ε_ι be the number specified in Proposition 2.9 and let $\varepsilon_{\mathrm{tub}} \in (0, \varepsilon_\iota)$. Then, by Proposition 2.9 (ii) there exists $C_{\iota,2} > 0$ such that

$$|\iota(x_\Sigma, t) - \iota(y_\Sigma, s)| \geq C_{\iota,2}^{-1}(|x_\Sigma - y_\Sigma| + |t - s|), \quad (x_\Sigma, t), (y_\Sigma, s) \in \Sigma \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}}).$$

Hence, $\iota \restriction \Sigma \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$ is injective, i.e. item (i) is true.

Next, we show assertion (ii). For this, we first define for $l \in \{1, \dots, p\}$ and $x' \in \mathbb{R}^{\theta-1}$ the matrices

$$\begin{aligned} M_l(x') &:= \big(\langle \partial_j \varkappa_l(x'), \partial_k \varkappa_l(x') \rangle\big)_{j,k \in \{1,\dots,\theta-1\}}, \\ H_l(x') &:= \big(\langle \partial_j \varkappa_l(x'), -\partial_k \nu_l(x') \rangle\big)_{j,k \in \{1,\dots,\theta-1\}}. \end{aligned}$$

Then, we have for $x' \in \mathbb{R}^{\theta-1}$ such that $\varkappa_l(x') = x_\Sigma \in \Sigma$ and $j, k \in \{1, \dots, \theta - 1\}$

$$\begin{aligned} H_l(x')[j,k] &= \langle \partial_j \varkappa_l(x'), W(x_\Sigma) \partial_k \varkappa_l(x') \rangle \\ &= \sum_{n=1}^{\theta-1} \langle \partial_j \varkappa_l(x'), L_l(x')[n,k] \partial_n \varkappa_l(x') \rangle \\ &= \sum_{n=1}^{\theta-1} M_l(x')[j,n] L_l(x')[n,k] \\ &= (M_l(x') L_l(x'))[j,k]. \end{aligned}$$

Moreover, using the definition of $\varkappa_l(x')$ one concludes $M_l(x') = I_{\theta-1} + \nabla\zeta_l(x')\nabla\zeta_l(x')^T$. The inverse of $M_l(x')$ is given by $I_{\theta-1} - (1 + |\nabla\zeta(x')|^2)^{-1}\nabla\zeta_l(x')\nabla\zeta_l(x')^T$. Hence,

$$L_l(x') = \big(I_{\theta-1} - (1 + |\nabla\zeta_l(x')|^2)^{-1}\nabla\zeta_l(x')\nabla\zeta_l(x')^T\big) H_l(x'). \tag{2.7}$$

Now, recall that $\det(I - \varepsilon W(x_\Sigma)) = \det(I_{\theta-1} - \varepsilon L_l(x'))$ for $x_\Sigma = \varkappa_l(x') \in \Sigma$. Expressing the determinant as the product of the eigenvalues one verifies the equation

$1-\det(I_{\theta-1}-\varepsilon L_l(x')) = \varepsilon P_l(\varepsilon)$, where P_l is a polynomial in ε with coefficients depending continuously on the entries of $L_l(x')$. Since $\zeta_l \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ by Definition 2.1, equation (2.7) and the definition of $H_l(x')$ imply $\sup_{l\in\{1,\ldots,p\},x'\in\varkappa_l^{-1}(\Sigma)}|L_l(x')| < \infty$. This shows that (ii) holds if $\varepsilon_{\text{tub}} > 0$ is chosen sufficiently small.

Finally, the claim in (iii) follows from [7, Proposition 2.6] since (i), (ii), and Proposition 2.9 (i) show that Σ satisfying Definition 2.1 also fulfils [7, Hypothesis 2.3]. $\square$

2.4 Bochner spaces

Bochner spaces play an essential role in this dissertation. They allow us to consider a certain class of integral operators, which are classically viewed as operators acting in $L^2(\Sigma \times (-1,1);\mathbb{C}^N)$, cf. [51, Section 3], as operators acting in the Bochner space $L^2((-1,1);L^2(\Sigma;\mathbb{C}^N))$. Moreover, by considering restrictions and extensions of these operators to $L^2((-1,1);H^r(\Sigma;\mathbb{C}^N))$ we can also incorporate Sobolev regularity in our analysis. With this motivation in mind, we study Bochner spaces and operators acting in Bochner spaces in the current section, which is an extended version of [14, Section 2.2] and is based on [43, Chapter 1].

In this section we always assume that $\mathcal{H}$ and $\mathcal{G}$ are separable Hilbert spaces, $(\mathscr{O},\mathscr{A},\lambda)$ and $(\mathscr{P},\mathscr{B},\mu)$ are measure spaces with σ-finite measures, and $\mathcal{L}(\mathcal{H},\mathcal{G})$ is the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$. Let us start by defining measurability for functions with values in Hilbert spaces.

Definition 2.13. *We call $f:\mathscr{O}\to\mathcal{H}$ (weakly) measurable if for all $\varphi\in\mathcal{H}$ the mapping $\mathscr{O}\ni t\mapsto (f(t),\varphi)_{\mathcal{H}}$ is measurable with respect to the measure λ. Furthermore, we call $F:\mathscr{O}\to\mathcal{L}(\mathcal{H},\mathcal{G})$ measurable if $\mathscr{O}\ni t\mapsto F(t)h$ is measurable for all $h\in\mathcal{H}$.*

Recall that a function $f:\mathscr{O}\to\mathcal{H}$ is (strongly) measurable if f is λ-a.e. the pointwise limit of simple functions, and that in the present situation both notions of measurability coincide due to Pettis theorem; see [43, Theorem 1.1.20]. Moreover, if $f:\mathscr{O}\to\mathcal{H}$ and $F:\mathscr{O}\to\mathcal{L}(\mathcal{H},\mathcal{G})$ are measurable, then the function $\mathscr{O}\ni t\mapsto F(t)f(t)\in\mathcal{G}$ is measurable; see [43, Proposition 1.1.28].

Definition 2.14. *We call a function $f:\mathscr{O}\to\mathcal{H}$ simple if there exist $n\in\mathbb{N}$, $\psi_1,\ldots,\psi_n\in\mathcal{H}$ and λ-measurable sets $\mathscr{A}_1,\ldots,\mathscr{A}_n\subset\mathscr{O}$ with finite measure such that $f=\sum_{l=1}^n \chi_{\mathscr{A}_l}\psi_l$, where $\chi_{\mathscr{A}_l}$ denotes the characteristic function of the set $\mathscr{A}_l$, $l\in\{1,\ldots,n\}$. In this case the Bochner integral of f is given by*

$$\int_{\mathscr{O}} f(t)\,d\lambda(t) := \sum_{l=1}^n \lambda(\mathscr{A}_l)\psi_l.$$

Moreover, we call a measurable function $f : \mathscr{O} \to \mathcal{H}$ Bochner integrable if there exists a sequence of simple functions $(f_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int_{\mathscr{O}} \|f - f_k\|_{\mathcal{H}} = 0.$$

In this case the sequence $(\int_{\mathscr{O}} f_k(s)\,ds)_{k\in\mathbb{N}}$ converges and we define the integral of f (which does not depend on the particular choice of the sequence $(f_k)_{k\in\mathbb{N}}$) by

$$\int_{\mathscr{O}} f(t)\,d\lambda(t) := \lim_{k\to\infty} \int_{\mathscr{O}} f_k(t)\,d\lambda(t).$$

By [43, Proposition 1.2.2] we have the following useful characterisation of Bochner integrability:

$$f \text{ is Bochner integrable} \iff f \text{ is measurable and } \int_{\mathscr{O}} \|f(t)\|\,d\lambda(t) < \infty. \tag{2.8}$$

Furthermore, it follows directly from the definition of the Bochner integral, that if $M \in \mathcal{L}(\mathcal{H},\mathcal{G})$ and $f : \mathscr{O} \to \mathcal{H}$ is integrable, then $Mf : \mathscr{O} \to \mathcal{G}$, $t \mapsto Mf(t)$, is integrable and there holds $M\int_{\mathscr{O}} f(t)\,d\lambda(t) = \int_{\mathscr{O}} Mf(t)\,d\lambda(t)$.

Many of the classical results from integration theory have natural extensions to the Bochner integration theory. Prominent examples of these extensions are Fubini's theorem and the dominated convergence theorem for Bochner integrals. They read as follows:

Proposition 2.15 (Fubini's theorem, [43, Proposition 1.2.7]). *Let $f : \mathscr{O} \times \mathscr{P} \to \mathcal{H}$ be a Bochner-integrable function. Then, the following statements hold:*

(i) *The function $t \mapsto f(t,s)$ is Bochner integrable for μ-a.e. $s \in \mathscr{P}$.*

(ii) *The function $s \mapsto f(t,s)$ is Bochner integrable for λ-a.e. $t \in \mathscr{O}$.*

(iii) *The functions $t \mapsto \int_{\mathscr{P}} f(t,s)\,d\mu(s)$ and $s \mapsto \int_{\mathscr{O}} f(t,s)\,d\lambda(t)$ are Bochner integrable and*

$$\begin{aligned}\int_{\mathscr{O}\times\mathscr{P}} f(t,s)\,d\lambda(t)d\mu(s) &= \int_{\mathscr{P}} \Big(\int_{\mathscr{O}} f(t,s)\,d\lambda(t)\Big)\,d\mu(s) \\ &= \int_{\mathscr{O}} \Big(\int_{\mathscr{P}} f(t,s)\,d\mu(s)\Big)\,d\lambda(t).\end{aligned}$$

Proposition 2.16 (Dominated convergence theorem, [43, Proposition 1.2.5]). *Let the functions $f_n : \mathscr{O} \to \mathcal{H}$, $n \in \mathbb{N}$, be Bochner integrable. If a function $f : \mathscr{O} \to \mathcal{H}$ as* *well as a nonnegative integrable function $g : \mathscr{O} \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n = f$ λ-a.e. and $\|f_n\|_{\mathcal{H}} \leq g$ λ-a.e. exist, then f is Bochner integrable and we have*

$$\lim_{n\to\infty} \int_{\mathscr{O}} \|f(t) - f_n(t)\|_{\mathcal{H}} \, d\lambda(t) = 0.$$

In particular,

$$\lim_{n\to\infty} \int_{\mathscr{O}} f_n(t) \, d\lambda(t) = \int_{\mathscr{O}} f(t) \, d\lambda(t).$$

Having stated these fundamental results regarding the Bochner integral, we turn to the definition of Bochner L^2-spaces.

Definition 2.17. *We define $L^2(\mathscr{O};\mathcal{H})$ as the space which contains all (equivalence classes of) λ-measurable functions $f : \mathscr{O} \to \mathcal{H}$ such that*

$$\int_{\mathscr{O}} \|f(t)\|_{\mathcal{H}}^2 \, d\lambda(t) < \infty.$$

Furthermore, we equip this space with the scalar product

$$\int_{\mathscr{O}} (f(t), g(t))_{\mathcal{H}} \, d\lambda(t), \qquad f, g \in L^2(\mathscr{O};\mathcal{H}).$$

It is not difficult to show that $L^2(\mathscr{O};\mathcal{H})$ is a Hilbert space; cf. [43, the comments below Definition 1.2.15]. The space $L^2(\mathscr{O};\mathcal{H})$ inherits many properties from $\mathcal{H}$. In this dissertation we are particularly interested in duality results, interpolation results, and in the case that $\mathcal{H}$ itself is a L^2-space, i.e. $\mathcal{H} = L^2(\mathscr{P};\mathcal{G})$. We summarize such results in the upcoming proposition.

Proposition 2.18. *Suppose that $L^2(\mathscr{O};\mathcal{H})$ and $L^2(\mathscr{O};\mathcal{G})$ are Bochner spaces. Then, the following holds:*

(i) *Functionals defined by*

$$L^2(\mathscr{O};\mathcal{H}) \ni g \mapsto \int_{\mathscr{O}} {}_{\mathcal{H}'}\langle f(t), g(t)\rangle_{\mathcal{H}} \, d\lambda(t), \qquad f \in L^2(\mathscr{O};\mathcal{H}'),$$

induce an isometric isomorphism between $L^2(\mathscr{O};\mathcal{H}')$ and the dual of $L^2(\mathscr{O};\mathcal{H})$, i.e.

$$L^2(\mathscr{O};\mathcal{H})' \simeq L^2(\mathscr{O};\mathcal{H}').$$

(ii) *If $\mathcal{G}$ is a Hilbert space such that $(\mathcal{H},\mathcal{G})$ is a compatible pair, cf. Section 2.1* (xiii), *then also $(L^2(\mathscr{O};\mathcal{H}), L^2(\mathscr{O};\mathcal{G}))$ is a compatible pair and*

$$L^2(\mathscr{O};[\mathcal{H},\mathcal{G}]_\tau) = \left[L^2(\mathscr{O};\mathcal{H}), L^2(\mathcal{O};\mathcal{G})\right]_\tau, \qquad \tau \in (0,1),$$

with equivalent norms.

(iii) *Let $\mathcal{H} = L^2(\mathscr{P};\mathcal{G})$. In this case the identification of $F \in L^2(\mathscr{O}\times\mathscr{P};\mathcal{G})$ with the function $f : t \mapsto F(t,\cdot)$ induces an isometric isomorphism between the spaces $L^2(\mathscr{O}\times\mathscr{P};\mathcal{G})$ and $L^2(\mathscr{O};\mathcal{H})$. Furthermore, if $f : \mathscr{O} \mapsto \mathcal{H}$ is Bochner integrable, then*

$$\left(\int_{\mathscr{O}} f(t)\,d\lambda(t)\right)(s) = \int_{\mathscr{O}} f(t)(s)\,d\lambda(t) \qquad \text{for } \mu\text{-a.e. } s \in \mathscr{P}.$$

Proof. The assertions (i), (ii) and (iii) follow from [43, Corollary 1.3.13 and Theorem 1.3.21], [43, Theorem 2.2.6 and Corollary C.4.2] and [43, Proposition 1.2.24 and Proposition 1.2.25], respectively. □

After providing elementary statements about Bochner spaces, we turn to operators in these spaces. We start by introducing the most simple and natural classes of operators. Let $\mathcal{Q} \in L^\infty(\mathscr{O})$ and $\mathcal{A} \in \mathcal{L}(\mathcal{H},\mathcal{G})$. In this case we define

$$\begin{aligned} &\mathcal{M}_{\mathcal{Q}} : L^2(\mathscr{O};\mathcal{H}) \to L^2(\mathscr{O};\mathcal{H}), \qquad &&(\mathcal{M}_{\mathcal{Q}}f)(t) := \mathcal{Q}(t)f(t),\\ &\mathcal{M}_{\mathcal{A}} : L^2(\mathscr{O};\mathcal{H}) \to L^2(\mathscr{O};\mathcal{G}), \qquad &&(\mathcal{M}_{\mathcal{A}}f)(t) := \mathcal{A}(f(t)). \end{aligned} \tag{2.9}$$

Note that the norms $\|\mathcal{M}_{\mathcal{Q}}\|_{L^2(\mathscr{O};\mathcal{H})\to L^2(\mathscr{O};\mathcal{H})}$ and $\|\mathcal{M}_{\mathcal{A}}\|_{L^2(\mathscr{O};\mathcal{H})\to L^2(\mathscr{O};\mathcal{G})}$ are equal to $\|\mathcal{Q}\|_{L^\infty(\mathcal{O})}$ and $\|\mathcal{A}\|_{\mathcal{H}\to\mathcal{H}}$, respectively. Moreover, from now on we identify $\mathcal{M}_{\mathcal{Q}}$ and $\mathcal{M}_{\mathcal{A}}$ with $\mathcal{Q}$ and $\mathcal{A}$. In the case that $\mathcal{O}$ is bounded the embedding

$$\mathfrak{J} : \mathcal{H} \to L^2(\mathscr{O};\mathcal{H}), \qquad (\mathfrak{J}\varphi)(t) := \varphi, \tag{2.10}$$

and its adjoint

$$\mathfrak{J}^* : L^2(\mathscr{O};\mathcal{H}) \to \mathcal{H}, \qquad \mathfrak{J}^*f = \int_{\mathcal{O}} f(t)\,dt \tag{2.11}$$

are well-defined and bounded. After introducing these simple operators, we turn to so-called *decomposable* operators. To do so, let us assume that

$$M \in L^\infty(\mathscr{O};\mathcal{L}(\mathcal{H},\mathcal{G})) := \big\{F : \mathscr{O} \to \mathcal{L}(\mathcal{H},\mathcal{G})\ \lambda\text{-measurable} : \|F\|_{\mathcal{H}\to\mathcal{G}} \in L^\infty(\mathscr{O})\big\}.$$

Then, we define the operator $\mathcal{M} : L^2(\mathscr{O};\mathcal{H}) \to L^2(\mathscr{O};\mathcal{G})$ through

$$(\mathcal{M}f)(t) := M(t)f(t) \quad \text{for } t \in \mathscr{O} \text{ and } f \in L^2(\mathscr{O};\mathcal{H}). \tag{2.12}$$

Such operators are generalizations of multiplication operators and are usually considered in the context of direct integrals, see for instance [27] or [64, Section XIII.16], where the operators $M(t)$, $t \in \mathscr{O}$, are called the fibers of $\mathcal{M}$. Similarly to before, we often identify $\mathcal{M}$ with M. Next, we summarize the main properties of these operators in the following proposition.

Proposition 2.19. *Let $M \in L^\infty(\mathscr{O};\mathcal{L}(\mathcal{H},\mathcal{G}))$ and $\mathcal{M}$ be defined by (2.12). Then, $\mathcal{M}$ is a well-defined bounded operator and its norm equals* $\operatorname{ess\,sup}_{t\in\mathscr{O}}\|M(t)\|_{\mathcal{H}\to\mathcal{G}}$. *Moreover, $\mathcal{M}$ is continuously invertible if and only if $M(t)$ is continuously invertible for λ-a.e. $t \in \mathscr{O}$ and $M^{-1} \in L^\infty(\mathscr{O};\mathcal{L}(\mathcal{G},\mathcal{H}))$. In this case there holds for every $g \in L^2(\mathscr{O};\mathcal{G})$ the equality $(\mathcal{M}^{-1}g)(t) = (M(t))^{-1}g(t)$ for λ-a.e. $t \in \mathscr{O}$.*

Proof. The proof follows from [27, Lemma 1.2 and Lemma 1.3]. □

Next, we consider operators induced by functions with values in the space of (unbounded) self-adjoint operators.

Proposition 2.20 ([64, Theorem XIII.85]). *Let A be a function mapping from $\mathscr{O}$ into the space of (unbounded) self-adjoint operators in $\mathcal{H}$, such that $(A+i)^{-1}$ is measurable. Moreover, let the operator $\mathcal{A}$ acting in $L^2(\mathscr{O};\mathcal{H})$ be defined by*

$$\begin{aligned}\operatorname{dom}\mathcal{A} &:= \big\{f \in L^2(\mathscr{O};\mathcal{H}) : f(t) \in \operatorname{dom}A(t) \text{ for } \lambda\text{-a.e. } t \in \mathscr{O} \text{ and the function} \\ &\qquad\qquad \text{defined by } t \mapsto A(t)f(t) \text{ is in } L^2(\mathscr{O};\mathcal{H})\big\}, \\ (\mathcal{A}f)(t) &:= A(t)f(t) \quad \text{for } \lambda\text{-a.e.} t \in \mathscr{O} \text{ and } f \in \operatorname{dom}\mathcal{A}.\end{aligned}$$

Then, the operator $\mathcal{A}$ is self-adjoint and $z \in \sigma(\mathcal{A})$ if and only if for all $\delta > 0$ the set $\{t \in \mathscr{O} : \sigma(A(t)) \cap (z-\delta, z+\delta) \neq \emptyset\}$ has positive measure.

As already mentioned in the beginning of this section we are particularly interested in the setting where $L^2(\mathscr{O};\mathcal{H}) = L^2((-1,1);H^r(S;\mathbb{C}^N))$, i.e. $\mathscr{O} = (-1,1)$, $\mathscr{A}$ is the corresponding σ-algebra of Lebesgue measurable sets on $(-1,1)$, λ is the classical Lebesgue measure and $\mathcal{H} = H^r(S;\mathbb{C}^N)$, $r \in [-2,2]$, where S is either $\mathbb{R}^{\theta-1}$ or $S \subset \mathbb{R}^\theta$ is a special C^2-surface as in Definition 2.1. In order to shorten notation, we set $\mathcal{B}^r(S) := L^2((-1,1,);H^r(S;\mathbb{C}^N))$ and use the conventions for the norm from Section 2.1 (xi) in this setting. We summarize important properties of these spaces in the following proposition.

Proposition 2.21. *Let $S = \mathbb{R}^{\theta-1}$ or $S \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1. Then, the following is true:*

(i) *If $\tau \in (0,1)$, $r_1, r_2 \in [-2,2]$ and $r = (1-\tau)r_1 + \tau r_2$, then*

$$\mathcal{B}^r(S) = \big[\mathcal{B}^{r_1}(S), \mathcal{B}^{r_2}(S)\big]_\tau$$

and the corresponding norms are equivalent.

(ii) *For $r \in [0,2]$ there exists an isometric isomorphism between the dual space of $\mathcal{B}^r(S)$ and $\mathcal{B}^{-r}(S)$, i.e.*

$$\mathcal{B}^r(S)' \simeq \mathcal{B}^{-r}(S).$$

(iii) *If* $0 \le r_1 \le r_2 \le 2$, *then* $\mathscr{B}^{r_2}(S)$ *is densely contained in* $\mathscr{B}^{r_1}(S)$.

Proof. Item (i) and (ii) follow from Proposition 2.18, the definition of $H^r(\Sigma;\mathbb{C}^N)$ in Section 2.2 and Proposition 2.2. It remains to prove (iii). We already know from Proposition 2.2 (ii) that $H^{r_2}(S;\mathbb{C}^N)$ is densely contained in $H^{r_1}(S;\mathbb{C}^N)$ if S is a special C^2-surface. If $S = \mathbb{R}^{\theta-1}$, then this is well known and can be found for example in [54, the text above eq. (3.22)]. Now, let $f \in \mathscr{B}^{r_1}(S)$ and $\delta > 0$. Then, according to [43, Lemma 1.2.19], there exists a simple function $f_\delta = \sum_{l=1}^n \chi_{\mathscr{A}_l}\psi_l$ with $\psi_1,\dots,\psi_n \in H^{r_1}(S;\mathbb{C}^N)$ and measurable sets $\mathscr{A}_1,\dots,\mathscr{A}_n$ with finite measure such that $\|f-f_\delta\|_{r_1} < \frac{\delta}{2}$. Moreover, we can choose $\widetilde{\psi}_l \in H^{r_2}(S;\mathbb{C}^N)$, $l \in \{1,\dots,n\}$, such that

$$\|\psi_l - \widetilde{\psi}_l\|_{H^{r_1}(S;\mathbb{C}^N)} < \frac{1}{2n\delta\|\chi_{\mathscr{A}_l}\|_{L^2(-1,1)}}.$$

Thus, $\widetilde{f}_\delta = \sum_{l=1}^n \chi_{\mathscr{A}_l}\widetilde{\psi}_l \in \mathscr{B}^{r_2}(S)$ and

$$\begin{aligned}\|f-\widetilde{f}_\delta\|_{r_1} &\le \|f-f_\delta\|_{r_1} + \|\widetilde{f}_\delta - f_\delta\|_{r_1}\\ &\le \frac{\delta}{2} + \sum_{l=1}^n \|(\widetilde{\psi}_l-\psi_l)\chi_{\mathscr{A}_l}\|_{r_1}\\ &= \frac{\delta}{2} + \sum_{l=1}^n \|\psi_l-\widetilde{\psi}_l\|_{H^{r_1}(S;\mathbb{C}^N)}\|\chi_{\mathscr{A}_l}\|_{L^2((-1,1))}\\ &< \delta,\end{aligned}$$

which finishes the proof. □

2.5 Norm resolvent convergence

In this section we study the convergence of unbounded self-adjoint operators via the concept of norm resolvent convergence. Throughout this section we assume that $\mathcal{I} \subset \mathbb{R}$ and $\mathcal{H}$ is a Hilbert space.

Definition 2.22. *Let* $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ *be a family of self-adjoint operators in* $\mathcal{H}$, $\omega_0 \in \overline{\mathcal{I}}$ *and* $\mathcal{A}$ *be a self-adjoint operator in* $\mathcal{H}$. *Then, we say that* $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ *(or simply* $\mathcal{A}_\omega$*) converges for* $\omega \to \omega_0$ *in the norm resolvent sense to* $\mathcal{A}$ *if for all* $z \in \mathbb{C}\setminus\mathbb{R}$

$$\big\|(\mathcal{A}_\omega - z)^{-1} - (\mathcal{A}-z)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} \to 0 \quad \text{for} \quad \omega \to \omega_0.$$

We start by stating a classical result which shows that $\mathcal{A}_\omega$ converges in the norm resolvent sense to $\mathcal{A}$ if $(\mathcal{A}_\omega - z_0)^{-1}$ converges to $(\mathcal{A}-z_0)^{-1}$ for only one $z_0 \in \rho(\mathcal{A})$.

Proposition 2.23. *Let $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ be a family of self-adjoint operators in $\mathcal{H}$, $\omega_0 \in \overline{\mathcal{I}}$ and $\mathcal{A}$ be a self-adjoint operator in $\mathcal{H}$. Moreover, assume that there exists a $z_0 \in \mathbb{C}$ such that $z_0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_\omega)$ if $|\omega - \omega_0|$ is sufficiently small and*

$$\big\|(\mathcal{A}_\omega - z_0)^{-1} - (\mathcal{A} - z_0)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} \to 0 \quad \text{for} \quad \omega \to \omega_0.$$

Then, $\mathcal{A}_\omega$ converges for $\omega \to \omega_0$ in the norm resolvent sense to $\mathcal{A}$.

Proof. The statement follows from [73, Satz 9.20 a)]. □

Next, we show that norm resolvent convergence is invariant with respect to bounded self-adjoint perturbations.

Proposition 2.24. *Let $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ be a family of self-adjoint operators in $\mathcal{H}$ which converges for $\omega \to \omega_0 \in \overline{\mathcal{I}}$ in the norm resolvent sense to the self-adjoint operator $\mathcal{A}$, $\mathcal{K}$ be a bounded self-adjoint operator in $\mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then,*

$$\begin{aligned}\big\|(\mathcal{A}_\omega + \mathcal{K} - z)^{-1} &- (\mathcal{A} + \mathcal{K} - z)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} \\ &\leq \big(1 + |\mathrm{Im}\, z|^{-1}\|\mathcal{K}\|_{\mathcal{H}\to\mathcal{H}}\big)^2 \big\|(\mathcal{A}_\omega - z)^{-1} - (\mathcal{A} - z)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} \quad \forall \omega \in \mathcal{I}.\end{aligned}$$

In particular, $\mathcal{A}_\omega + \mathcal{K}$ converges for $\omega \to \omega_0$ in the norm resolvent sense to $\mathcal{A} + \mathcal{K}$.

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then,

$$\begin{aligned}&(\mathcal{A}_\omega + \mathcal{K} - z)^{-1} - (\mathcal{A} + \mathcal{K} - z)^{-1} \\ &\quad = \big(I - (\mathcal{A} + \mathcal{K} - z)^{-1}\mathcal{K}\big)\big((\mathcal{A}_\omega - z)^{-1} - (\mathcal{A} - z)^{-1}\big)\big(I - \mathcal{K}(\mathcal{A}_\omega + \mathcal{K} - z)^{-1}\big)\end{aligned}$$

and therefore

$$\begin{aligned}\big\|(\mathcal{A}_\omega + \mathcal{K} - z)^{-1} &- (\mathcal{A} + \mathcal{K} - z)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} \\ &\leq \big(1 + |\mathrm{Im}\, z|^{-1}\|\mathcal{K}\|_{\mathcal{H}\to\mathcal{H}}\big)^2 \big\|(\mathcal{A}_\omega - z)^{-1} - (\mathcal{A} - z)^{-1}\big\|_{\mathcal{H}\to\mathcal{H}}\end{aligned}$$

converges to zero for $\omega \to \omega_0$. □

In the next proposition we summarize important spectral implications of the norm resolvent convergence.

Proposition 2.25. *Let $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ be a family of self-adjoint operators in $\mathcal{H}$ which converges for $\omega \to \omega_0 \in \overline{\mathcal{I}}$ in the norm resolvent sense to the self-adjoint operator $\mathcal{A}$. Then, there holds the following:*

(i) $\lim_{\omega\to\omega_0} \sigma(\mathcal{A}_\omega) = \sigma(\mathcal{A})$ *is valid; this notation means on the one hand that the limit of every convergent sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \sigma(\mathcal{A}_{\omega_n})$ for a $\omega_n \in \mathcal{I}$ and $\omega_n \overset{n\to\infty}{\to} \omega_0$ is in $\sigma(\mathcal{A})$. On the other hand this means also that every $\lambda \in \sigma(\mathcal{A})$ is the limit of such a sequence.*

(ii) $\lim_{\omega \to \omega_0} \sigma_{\text{ess}}(\mathcal{A}_\omega) = \sigma_{\text{ess}}(\mathcal{A})$ *in the same sense as described in* (i).

(iii) *If* $a, b \in \rho(\mathcal{A})$ *with* $a < b$, *then* $a, b \in \rho(\mathcal{A}_\omega)$ *for* $|\omega - \omega_0|$ *sufficiently small and there holds*

$$\|P_{\mathcal{A}_\omega}(a,b) - P_{\mathcal{A}}(a,b)\|_{\mathcal{H}\to\mathcal{H}} \to 0 \quad \text{for} \quad \omega \to \omega_0,$$

where $P_{\mathcal{A}_\omega}$ *and* $P_{\mathcal{A}}$ *denote the spectral measures of* $\mathcal{A}_\omega$ *and* $\mathcal{A}$, *respectively.*

Proof. See [73, Satz 9.24]. □

Next, we present two consequences of the previous proposition.

Proposition 2.26. *Let* $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ *be a family of self-adjoint operators in* $\mathcal{H}$ *which converges for* $\omega \to \omega_0 \in \overline{\mathcal{I}}$ *in the norm resolvent sense to the self-adjoint operator* $\mathcal{A}$ *and assume that* $\mathcal{A}$ *has at least* $M \in \mathbb{N}$ *discrete eigenvalues counted with multiplicity. Then,* $\mathcal{A}_\omega$ *has also at least* M *discrete eigenvalues counted with multiplicity for sufficiently small* $|\omega - \omega_0|$.

Proof. The proof is based on the proofs of [8, Proposition 5.5] and [10, Theorem 2.7]. Let $\lambda_1, \dots, \lambda_{n_M}$, $n_M \in \mathbb{N}$, be distinct discrete eigenvalues of $\mathcal{A}$ such that the sum of their multiplicities equals M. Since, $\lambda_1, \dots, \lambda_{n_M}$, are discrete eigenvalues, there exist $a_1, \dots, a_{n_M} \in \mathbb{R}$ and $b_1, \dots, b_{n_M} \in \mathbb{R}$ such that the intervals (a_j, b_j), $j \in \{1, \dots, n_M\}$, are pairwise disjoint, $\lambda_j \in (a_j, b_j)$ and $[a_j, b_j]\backslash\{\lambda_j\} \subset \rho(\mathcal{A})$ for all $j \in \{1, \dots, n_M\}$. In this case Proposition 2.25 implies that the spectral projections $P_{\mathcal{A}_\omega}(a_j, b_j)$ converge in the operator norm to $P_{\mathcal{A}}(a_j, b_j)$ for $\omega \to \omega_0$. Hence,

$$P_{\mathcal{A}_\omega}\Big(\bigcup_{j=1}^{n_M}(a_j,b_j)\Big) = \sum_{j=1}^{n_M} P_{\mathcal{A}_\omega}(a_j,b_j) \xrightarrow{\omega\to\omega_0} \sum_{j=1}^{n_M} P_{\mathcal{A}}(a_j,b_j) = P_{\mathcal{A}}\Big(\bigcup_{j=1}^{n_M}(a_j,b_j)\Big)$$

in the operator norm. Thus, [73, Satz 2.58] shows that the dimensions of the ranges of $P_{\mathcal{A}_\omega}\big(\bigcup_{j=1}^{n_M}(a_j,b_j)\big)$ and $P_{\mathcal{A}}\big(\bigcup_{j=1}^{n_M}(a_j,b_j)\big)$ coincide for sufficiently small $|\omega-\omega_0|$. Furthermore, there holds by construction $\dim\big(\operatorname{ran} P_{\mathcal{A}}\big(\bigcup_{j=1}^{n_M}(a_j,b_j)\big)\big) = M$. Hence, the dimension of the range of $P_{\mathcal{A}_\omega}\big(\bigcup_{j=1}^{n_M}(a_j,b_j)\big)$ equals M if $|\omega-\omega_0|$ is small enough. As $M < \infty$, [66, Proposition 8.11 (iv)] implies $\bigcup_{j=1}^{n_M}(a_j,b_j) \cap \sigma_{\text{ess}}(\mathcal{A}_\omega) = \emptyset$ for sufficiently small $|\omega-\omega_0|$. Hence, all M eigenvalues in $\bigcup_{j=1}^{n_M}(a_j,b_j)$ are discrete eigenvalues for sufficiently small $|\omega-\omega_0|$. □

Proposition 2.27. *Let* $(\mathcal{A}_\omega)_{\omega\in\mathcal{I}}$ *be a family of self-adjoint operators in* $\mathcal{H}$ *which converges for* $\omega \to \omega_0 \in \overline{\mathcal{I}}$ *in the norm resolvent sense to the self-adjoint operator* $\mathcal{A}$ *and assume that* $K \subset \mathbb{R}$ *is a compact set. Then, there holds the following:*

(i) *If* $\sigma(\mathcal{A}) \cap K = \emptyset$, *then* $\sigma(\mathcal{A}_\omega) \cap K = \emptyset$ *for sufficiently small* $|\omega-\omega_0|$.

(ii) *If* $\sigma_{\text{ess}}(\mathcal{A}) \cap K = \emptyset$, *then* $\sigma_{\text{ess}}(\mathcal{A}_\omega) \cap K = \emptyset$ *for sufficiently small* $|\omega-\omega_0|$.

Proof. We prove (i) by contraposition; the proof of (ii) follows along the same lines. Let us assume that there exist sequences $(\omega_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ such that $\omega_n \overset{n\to\infty}{\longrightarrow} \omega_0$ and $\lambda_n \in \sigma(\mathcal{A}_{\omega_n}) \cap K$ for all $n \in \mathbb{N}$. Moreover, as K is compact it is no restriction to assume that $(\lambda_n)_{n\in\mathbb{N}}$ converges to some $\lambda \in K$. Since $\lambda_n \in \sigma(\mathcal{A}_n)$ for all $n \in \mathbb{N}$, Proposition 2.25 (i) implies $\lambda \in \sigma(\mathcal{A})$, which yields $\lambda \in \sigma(\mathcal{A})\cap K$, i.e. $\sigma(\mathcal{A})\cap K \neq \emptyset$. $\square$

2.6 Invertibility of bounded operators

We provide in this short section results regarding the invertibility of bounded operators. Throughout this section $\mathcal{H}$ denotes a Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded operators mapping from $\mathcal{H}$ to $\mathcal{H}$.

Proposition 2.28. *Let $\mathcal{K}, \mathcal{T} \in \mathcal{L}(\mathcal{H})$ and assume $\mathcal{T}^{-1} \in \mathcal{L}(\mathcal{H})$ as well as*

$$\left\|\mathcal{T}^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \|\mathcal{K}\|_{\mathcal{H}\to\mathcal{H}} < 1.$$

Then, $(\mathcal{T}+\mathcal{K})^{-1} \in \mathcal{L}(\mathcal{H})$ and

$$\left\|(\mathcal{T}+\mathcal{K})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{\left\|\mathcal{T}^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}}{1-\left\|\mathcal{T}^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}\|\mathcal{K}\|_{\mathcal{H}\to\mathcal{H}}}.$$

Proof. This is a well-known result which follows for example from [44, Chapter IV, Theorem 1.16 and Remark 1.17]. $\square$

Next, we provide a result which is known as Jacobson's lemma.

Proposition 2.29. *Let $\mathcal{T}, \mathcal{K} \in \mathcal{L}(\mathcal{H})$. Then, $\rho(\mathcal{K}\mathcal{T}) \setminus \{0\} = \rho(\mathcal{T}\mathcal{K}) \setminus \{0\}$ and for $z \in \rho(\mathcal{K}\mathcal{T}) \setminus \{0\}$ the formulas*

$$\begin{aligned}(\mathcal{T}\mathcal{K}-z)^{-1}\mathcal{T} &= \mathcal{T}(\mathcal{K}\mathcal{T}-z)^{-1}\\ (\mathcal{T}\mathcal{K}-z)^{-1} &= \frac{1}{z}\big(\mathcal{T}(\mathcal{K}\mathcal{T}-z)^{-1}\mathcal{K}-I\big)\end{aligned}$$

are valid.

Proof. The result about the resolvent sets can be found in [57, Proposition 2.1.8]. Furthermore, the formulas can be verified by applying $\mathcal{T}\mathcal{K}-z$ and using the identity $(\mathcal{T}\mathcal{K}-z)\mathcal{T} = \mathcal{T}(\mathcal{K}\mathcal{T}-z)$. $\square$

Finally, we present an invertibility result tailored to later applications in this thesis.

Proposition 2.30. *Let* $\mathcal{A}, \mathcal{A}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{T} \in \mathcal{L}(\mathcal{H})$*, and assume* $\mathcal{A}\mathcal{T} = I + \mathcal{K}_1 + \mathcal{K}_2$*,* $\mathcal{A}_0^{-1} \in \mathcal{L}(\mathcal{H})$ *and*

$$\left\|\mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right\|_{\mathcal{H}\to\mathcal{H}} < 1. \tag{2.13}$$

Then, a bounded right inverse of $\mathcal{A}$ *is given by*

$$\left(\mathcal{T} - \mathcal{A}_0^{-1}\mathcal{K}_2\right)\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)^{-1}.$$

In particular, this right inverse of $\mathcal{A}$ *can be estimated by the expression*

$$\frac{\left\|\mathcal{T} - \mathcal{A}_0^{-1}\mathcal{K}_2\right\|_{\mathcal{H}\to\mathcal{H}}}{1 - \left\|\mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right\|_{\mathcal{H}\to\mathcal{H}}}. \tag{2.14}$$

Proof. By (2.13) we can apply Proposition 2.28 to $I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2$. Hence, $\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)^{-1} \in \mathcal{L}(\mathcal{H})$ and its norm is bounded by

$$\frac{1}{1 - \left\|\mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right\|_{\mathcal{H}\to\mathcal{H}}}.$$

Next, we calculate the product

$$\begin{aligned}
&\mathcal{A}\left(\mathcal{T} - \mathcal{A}_0^{-1}\mathcal{K}_2\right)\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\ \right)^{-1} \\
&\quad = \left(I + \mathcal{K}_1 + \mathcal{K}_2 - \mathcal{A}\mathcal{A}_0^{-1}\mathcal{K}_2\right)\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)^{-1} \\
&\quad = \left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)^{-1} \\
&\quad = I.
\end{aligned}$$

Thus, $\left(\mathcal{T} - \mathcal{A}_0^{-1}\mathcal{K}_2\right)\left(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\right)^{-1}$ is a right inverse of $\mathcal{A}$ and its norm can be estimated by (2.14). $\square$

3 The free Dirac operator and perturbed Dirac operators

As mentioned in the introduction, Dirac operators model spin 1/2 particles subject to an external field which is modelled by a potential. In this chapter we provide elementary results for Dirac operators with various potentials. We start by dealing with the free Dirac operator, where the potential is set to zero, in Section 3.1. Then, we turn to Dirac operators with regular potentials, i.e. potentials which can be described by bounded operators. Lastly, in Section 3.3, we deal with Dirac operators with δ-shell potentials, which are potentials that are only supported on a $(\theta - 1)$-dimensional hypersurface in $\mathbb{R}^\theta$.

3.1 The free Dirac operator and associated integral operators

In this section we lay the foundation for the study of Dirac operators. We start by defining the free Dirac operator. Afterwards, we study potential and boundary integral operators induced by the kernel of the resolvent of the free Dirac operator. Let us also mention that this section is based on [14, Section 2.3].

Before we introduce the free Dirac operator we define the so-called Dirac matrices.

Definition 3.1. *The Pauli spin matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With their help the Dirac matrices $\alpha_1, \dots, \alpha_\theta, \beta \in \mathbb{C}^{N\times N}$ are defined for $\theta = 2$ by

$$\alpha_1 := \sigma_1, \quad \alpha_2 := \sigma_2, \quad \beta := \sigma_3,$$

and for $\theta = 3$ by

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3 \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I_2 is the 2×2-identity matrix. We often make use of the abbreviations

$$\alpha \cdot \nabla := \sum_{j=1}^{\theta} \alpha_j \partial_j \quad \text{and} \quad \alpha \cdot x := \sum_{j=1}^{\theta} \alpha_j x_j, \quad x = (x_1, \dots, x_\theta) \in \mathbb{C}^\theta.$$

The Dirac matrices fulfil the useful relations

$$\alpha_j\alpha_l + \alpha_l\alpha_j = 2\delta_{jl} \quad \text{and} \quad \alpha_j\beta + \beta\alpha_j = 0 \quad \forall j, l \in \{1, \dots, \theta\}, \tag{3.1}$$

where δ_{jl} denotes the Kronecker delta. Using the self-adjointness of the Dirac matrices and (3.1) implies that for $x \in \mathbb{R}^\theta$ the matrix $\alpha \cdot x$ is self-adjoint and $(\alpha \cdot x)^2 = |x|^2 I_N$.

Now, we are in the position to define the free Dirac operator.

Definition 3.2. *Let $m \in \mathbb{R}$. Then, the free Dirac operator H is defined by*

$$H = -i(\alpha \cdot \nabla) + m\beta, \qquad \operatorname{dom} H = H^1(\mathbb{R}^\theta; \mathbb{C}^N).$$

In the next proposition we summarize important properties of the free Dirac operator.

Proposition 3.3. *The free Dirac operator H is self-adjoint and the following is true:*

(i) $\sigma(H) = \sigma_{\mathrm{ess}}(H) = (-\infty, -|m|] \cup [|m|, \infty)$.

(ii) *For $z \in \rho(H) = \mathbb{C} \setminus (-\infty, -|m|] \cup [|m|, \infty)$ the resolvent $R(z) := (H - z)^{-1}$ can be expressed by*

$$R(z)u(x) = \int_{\mathbb{R}^\theta} G_z(x - y)u(y)\, dy, \qquad u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N),\ x \in \mathbb{R}^\theta, \tag{3.2}$$

where G_z is given for $\theta = 2$ and $x \in \mathbb{R}^2 \setminus \{0\}$ by

$$\begin{aligned} G_z(x) = \frac{\sqrt{z^2 - m^2}}{2\pi} &K_1\big(-i\sqrt{z^2 - m^2}|x|\big)\frac{\alpha \cdot x}{|x|} \\ &+ \frac{1}{2\pi}K_0\big(-i\sqrt{z^2 - m^2}|x|\big)\,(m\beta + zI_2) \end{aligned} \tag{3.3}$$

and for $\theta = 3$ and $x \in \mathbb{R}^3 \setminus \{0\}$ by

$$G_z(x) = \left(zI_4 + m\beta + i\Big(1 - i\sqrt{z^2 - m^2}|x|\Big)\frac{\alpha \cdot x}{|x|^2}\right)\frac{e^{i\sqrt{z^2 - m^2}|x|}}{4\pi|x|}. \tag{3.4}$$

The expressions K_0 and K_1 denote the modified Bessel functions of the second kind of order zero and one, respectively.

(iii) *$R(z)$ acts as a bounded operator from $H^r(\mathbb{R}^\theta; \mathbb{C}^N)$ to $H^{r+1}(\mathbb{R}^\theta; \mathbb{C}^N)$ for all $r \in \mathbb{R}$.*

Proof. All statements besides (ii) can be proven in a straight forward manner by using the Fourier transform; see for instance [18, Section 2] for $\theta = 2$ and [71, Theorem 1.1] for $\theta = 3$. The formulas for G_z can be found in [13, eq. (3.2)], [16, eq. (1.19)] or [71, eq. (1.263)]. $\square$

Next, we provide helpful estimates for the integral kernel G_z.

Proposition 3.4. *Let $z \in \rho(H)$ and G_z be given by (3.3)–(3.4). Then, there exist $C_{G,1} = C_{G,1}(m, z) > 0$ and $C_{G,2} = C_{G,2}(m, z) > 0$ such that for all $x \in \mathbb{R}^\theta \setminus \{0\}$ and $j, k \in \{1, \dots, \theta\}$*

$$\begin{aligned} |G_z(x)| &\leq C_{G,1}|x|^{1-\theta}e^{-C_{G,2}|x|}, \\ |\partial_j G_z(x)| &\leq C_{G,1}|x|^{-\theta}e^{-C_{G,2}|x|}, \\ |\partial_k\partial_j G_z(x)| &\leq C_{G,1}|x|^{-1-\theta}e^{-C_{G,2}|x|}. \end{aligned}$$

Proof. We start by noticing that $G_z = (-i(\alpha \cdot \nabla) + m\beta + zI_N)g_z$, where

$$g_z(x) = \begin{cases} \frac{1}{2\pi}K_0(-i\sqrt{z^2-m^2}|x|), & \theta = 2, \\ \frac{1}{4\pi}\frac{e^{i\sqrt{z^2-m^2}|x|}}{|x|}, & \theta = 3, \end{cases}$$

for $x \in \mathbb{R}^\theta \setminus \{0\}$. This follows from a direct calculation and from using for the modified Bessel functions the rule $K_0' = -K_1$; see [55, §10.29 (i)]. Next, we introduce for $l \in \mathbb{N}_0$ the set of functions

$$\begin{aligned} P_l := \Big\{ p \in C(\mathbb{R}^\theta \setminus \{0\}) : p(x) = \sum_{j=1}^{m} a_j x^{\gamma_j}|x|^{-k_j} \text{ with } m \in \mathbb{N}, a_j \in \mathbb{C}, \\ k_j \in \mathbb{N}_0 \text{ and } \gamma_j = (\gamma_{j,1}, \dots, \gamma_{j,\theta}) \in \mathbb{N}_0^\theta \\ \text{such that } -l \leq \gamma_{j,1} + \dots + \gamma_{j,\theta} - k_j \leq 0 \Big\}. \end{aligned}$$

Note that for $p \in P_l$ exists a $C > 0$ such that

$$|p(x)| \leq C(1 + |x|^{-l}) \qquad \forall x \in \mathbb{R}^\theta \setminus \{0\}. \tag{3.5}$$

Now, let $\lambda \in \mathbb{N}_0^\theta$ be a multi-index with $\lambda_1 + \dots + \lambda_\theta = n$, $n \in \mathbb{N}$. The chain rule, the product rule and induction show that if $\theta = 2$, then there exist functions $p_{\lambda,l} \in P_l$, $l \in \{0, \dots, n-1\}$, such that for $x \in \mathbb{R}^2 \setminus \{0\}$

$$\partial^\lambda g_z(x) := \partial_{\lambda_1} \cdots \partial_{\lambda_n} g_z(x) = \sum_{l=0}^{n-1} K_0^{(n-l)}(-i\sqrt{z^2-m^2}|x|)p_{\lambda,l}(x), \tag{3.6}$$

and if $\theta = 3$, then there exists a function $p_\lambda \in P_{n+1}$ such that for $x \in \mathbb{R}^3 \setminus \{0\}$

$$\partial^\lambda g_z(x) = p_\lambda(x)e^{i\sqrt{z^2-m^2}|x|}. \tag{3.7}$$

By the well-known rules for the derivatives and the asymptotic expansions of the modified Bessel functions from [55, §10.25 (ii), §10.27, §10.29 and §10.30 (i)], there exists an $R > 0$ such that for all $x \in \mathbb{R}^2 \setminus \{0\}$ with $|x| \leq R$ one has

$$\big|K_0^{(k)}\big(-i\sqrt{z^2-m^2}|x|\big)\big| \leq \begin{cases} C|\log|x||, & k = 0, \\ C|x|^{-k}, & k \in \mathbb{N}, \end{cases}$$

and for $|x| > R$

$$\left|K_0^{(k)}\left(-i\sqrt{z^2-m^2}|x|\right)\right| \leq Ce^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, \quad k \in \mathbb{N}_0.$$

In particular, there exists a $C > 0$ such that for all $x \in \mathbb{R}^2 \setminus \{0\}$

$$\left|K_0^{(k)}(-i\sqrt{z^2-m^2}|x|)\right| \leq \begin{cases} C(1+|x|^{-1})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, & k=0, \\ C(1+|x|^{-k})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, & k \in \mathbb{N}. \end{cases} \tag{3.8}$$

Inserting this and (3.5) into (3.6) and (3.7) yields

$$\left|\partial^\lambda g_z(x)\right| \leq C(1+|x|^{-n+(2-\theta)})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|} \qquad \forall x \in \mathbb{R}^\theta \setminus \{0\}.$$

Moreover, using this result and (3.8) for $k=0$ yields

$$\begin{aligned} |G_z(x)| &\leq C(1+|x|^{1-\theta})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, \\ |\partial_j G_z(x)| &\leq C(1+|x|^{-\theta})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, \\ |\partial_k\partial_j G_z(x)| &\leq C(1+|x|^{-1-\theta})e^{-\mathrm{Im}\sqrt{z^2-m^2}|x|}, \end{aligned}$$

for all $x \in \mathbb{R}^\theta \setminus \{0\}$ and $j,k \in \{1,\dots,\theta\}$, where $C = C(m,z) > 0$ is a constant which only depends on m and z. Thus, the estimates of the proposition are valid if one chooses $C_{G,2} \in (0, \mathrm{Im}\sqrt{z^2-m^2})$ and

$$C_{G,1} = \max_{x \in \mathbb{R}^\theta \setminus \{0\}, l \in \{1-\theta,-\theta,-1-\theta\}} C\frac{1+|x|^l}{|x|^l}e^{-(\mathrm{Im}\sqrt{m^2-z^2}-C_{G,2})|x|} < \infty. \qquad \square$$

Having discussed the free Dirac operator and its resolvent, we define potential and boundary integral operators with the help of G_z. We start with the potential operator.

Definition 3.5. *Let $z \in \rho(H)$, G_z be given by (3.3)–(3.4) and $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1. Then, we introduce the potential operator $\Phi_z : L^2(\Sigma;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ by*

$$\Phi_z\varphi(x) := \int_\Sigma G_z(x-y_\Sigma)\varphi(y_\Sigma)\,d\sigma(y_\Sigma), \qquad \varphi \in L^2(\Sigma;\mathbb{C}^N),\ x \in \mathbb{R}^\theta,$$

which is well-defined and bounded by Proposition 3.4 and [4, Lemma 2.1].

Next, we study the mapping properties of this operator in detail. Note that these properties are well known for the case that Σ is compact; see e.g. [12, Propoisition 4.2] or [16, Theorem 4.3].

Proposition 3.6. *Let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ and let Φ_z be given by Definition 3.5. Then, the following is true:*

(i) *For any $r \in [0, \frac{1}{2}]$ the operator Φ_z gives rise to a bounded operator*

$$\Phi_z : H^r(\Sigma; \mathbb{C}^N) \to H^{r+1/2}(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N) = H^{r+1/2}(\Omega_+; \mathbb{C}^N) \oplus H^{r+1/2}(\Omega_-; \mathbb{C}^N).$$

(ii) *For $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ one has $(-i(\alpha \cdot \nabla) + m\beta - zI_N)(\Phi_z \varphi)_\pm = 0$.*

(iii) *The adjoint $\Phi_z^* : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to L^2(\Sigma; \mathbb{C}^N)$ of Φ_z acts on $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ as*

$$\Phi_z^* u(x_\Sigma) = \int_{\mathbb{R}^\theta} G_{\overline{z}}(x_\Sigma - y) u(y)\, dy = \boldsymbol{t}_\Sigma R(\overline{z}) u(x_\Sigma), \qquad x_\Sigma \in \Sigma, \tag{3.9}$$

and Φ_z^* gives rise to a bounded operator $\Phi_z^* : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$.

Before we prove the assertion we shortly introduce auxiliary operators and make some remarks. Although we are not using boundary triples in the proof explicitly, it is heavily inspired by [16, Theorem 4.3], where boundary triple techniques were used to show the assertion in the case that Σ is compact. To use such techniques, it is helpful to introduce the bounded operators $\Gamma_0, \Gamma_1 : H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$ which are defined by $\Gamma_0 := i(\alpha \cdot \nu)(\boldsymbol{t}_\Sigma^+ - \boldsymbol{t}_\Sigma^-)$ and $\Gamma_1 := \frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)$, where ν is the unit outward normal vector field of Σ described in Definition 2.1, and $\boldsymbol{t}_\Sigma^+$ and $\boldsymbol{t}_\Sigma^-$ are the trace operators from Remark 2.4. Moreover, let

$$Xu := (-i(\alpha\cdot\nabla)+m\beta)u_+ \oplus (-i(\alpha\cdot\nabla)+m\beta)u_-, \quad \operatorname{dom} X := H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N). \tag{3.10}$$

Readers familiar with boundary triples may notice that $\{L^2(\Sigma; \mathbb{C}^N), \Gamma_0, \Gamma_1\}$ constitutes a quasi boundary triple for $\overline{X}$; cf. [16, Section 4.2]. By Corollary 2.5 the just introduced operators fulfil for all $u, v \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ the equation

$$(Xu, v)_{L^2(\mathbb{R}^\theta; \mathbb{C}^N)} - (u, Xv)_{L^2(\mathbb{R}^\theta; \mathbb{C}^N)} = (\Gamma_1 u, \Gamma_0 v)_{L^2(\Sigma; \mathbb{C}^N)} - (\Gamma_0 u, \Gamma_1 v)_{L^2(\Sigma; \mathbb{C}^N)}. \tag{3.11}$$

This shows that $X \restriction \ker \Gamma_0$ is symmetric. Moreover, $H \subset X \restriction \ker \Gamma_0$. Thus, as H is self-adjoint, $H = X \restriction \ker \Gamma_0$.

Proof of Proposition 3.6. First, Fubini's theorem shows that the representation of Φ_z^* in (3.9) is valid. Hence, the mapping properties of $\boldsymbol{t}_\Sigma$ and $R(z)$ prove assertion (iii).

To verify item (i), we note that by (iii) and antiduality Φ_z has the bounded extension

$$\widetilde{\Phi}_z := (\Phi_z^*)' : H^{-1/2}(\Sigma; \mathbb{C}^N) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N) = H^0(\Omega_+; \mathbb{C}^N) \oplus H^0(\Omega_-; \mathbb{C}^N). \tag{3.12}$$

Next, we show the statement for $r = \frac{1}{2}$. If we manage to do that, then the claim for $r \in [0, \frac{1}{2})$ follows from (3.12) and interpolation.

To prove the claim for $r = \frac{1}{2}$ we note that with $X \upharpoonright \ker \Gamma_0 = H$ one can show for $z \in \rho(H)$ the direct sum decomposition

$$\operatorname{dom} X = \operatorname{dom} H \dot{+} \ker(X - z) = \ker \Gamma_0 \dot{+} \ker(X - z),$$

which allows us to define the auxiliary operator

$$\widehat{\Phi}_z := (\Gamma_0 \upharpoonright \ker(X - z))^{-1}. \tag{3.13}$$

Note that the properties of the trace operator in Proposition 2.3 imply the equality $\operatorname{ran} \Gamma_0 = H^{1/2}(\Sigma; \mathbb{C}^N)$ and we also have $\operatorname{dom} X = H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. Thus, $\widehat{\Phi}_z$ is a linear operator from $H^{1/2}(\Sigma; \mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. Next, we show that $\widehat{\Phi}_z$ is a restriction of Φ_z. To see this, we observe for $v \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$, and $u = R(\overline{z})v = (H - \overline{z})^{-1}v \in \operatorname{dom} H = \ker \Gamma_0$ with the help of (3.11) that

$$\begin{aligned}
(\widehat{\Phi}_z\varphi, v)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} &= \big(\widehat{\Phi}_z\varphi, (H - \overline{z})u\big)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= (\widehat{\Phi}_z\varphi, Hu)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} - (z\widehat{\Phi}_z\varphi, u)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= (\widehat{\Phi}_z\varphi, Xu)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} - (X\widehat{\Phi}_z\varphi, u)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= -(\Gamma_1\widehat{\Phi}_z\varphi, \Gamma_0 u)_{L^2(\Sigma;\mathbb{C}^N)} + (\Gamma_0\widehat{\Phi}_z\varphi, \Gamma_1 u)_{L^2(\Sigma;\mathbb{C}^N)} \\
&= (\varphi, \Gamma_1 R(\overline{z})v)_{L^2(\Sigma;\mathbb{C}^N)} = (\varphi, (\Phi_z)^* v)_{L^2(\Sigma;\mathbb{C}^N)} \\
&= (\Phi_z\varphi, v)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)}.
\end{aligned}$$

Since this is true for all $v \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, we conclude $\widehat{\Phi}_z\varphi = \Phi_z\varphi$; hence $\widehat{\Phi}_z$ is the restriction of Φ_z to $H^{1/2}(\Sigma; \mathbb{C}^N)$. In particular, $\Phi_z\varphi \in \ker(X - z)$ by (3.13), which yields item (ii). Eventually, we show that this restriction of Φ_z is bounded from $H^{1/2}(\Sigma; \mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. To see this, we prove that $\widehat{\Phi}_z$ is closed with respect to these spaces. But this follows from the L^2-boundedness of Φ_z and the fact that $H^{1/2}(\Sigma; \mathbb{C}^N)$ and $H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ are continuously embedded in $L^2(\Sigma; \mathbb{C}^N)$ and $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, respectively. Thus, the closed graph theorem shows that

$$\widehat{\Phi}_z = \Phi_z \upharpoonright H^{1/2}(\Sigma; \mathbb{C}^N) : H^{1/2}(\Sigma; \mathbb{C}^N) \to H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N) = H^1(\Omega_+; \mathbb{C}^N) \oplus H^1(\Omega_-; \mathbb{C}^N)$$

is bounded, which finishes the proof. $\square$

Finally, we define a boundary integral operator associated with the free Dirac operator.

Definition 3.7. *Let $z \in \rho(H)$ and $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1. Then, we define operator $\mathcal{C}_z : L^2(\Sigma; \mathbb{C}^N) \to L^2(\Sigma; \mathbb{C}^N)$ as the unique bounded extension of*

$$H^{1/2}(\Sigma; \mathbb{C}^N) \ni f \mapsto \frac{1}{2}(\boldsymbol{t}_\Sigma^+ + \boldsymbol{t}_\Sigma^-)\Phi_z f.$$

We prove in Proposition 3.8 that $\mathcal{C}_z$ is well-defined. We remark that this operator could also be introduced as a strongly integral operator on Σ with the kernel G_z; cf. for instance [4, Lemma 3.3] or [16, eq. (4.5)]. However, in our case this abstract definition is more convenient. In particular, we do not have to deal with the existence of principal value integrals and corresponding difficulties.

Next, we summarize important properties of $\mathcal{C}_z$. Similar as in the case of Φ_z, these properties are well known in various settings; see [4, Lemma 3.3] and [16, eq. (4.10) and Theorem 4.3] if Σ is compact, and [19, Lemma 2.1 and Corollary 2.1] if Σ is a plane with a compact perturbation.

Proposition 3.8. *Let $z \in \rho(H) = \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ and let $\mathcal{C}_z$ be given by Definition 3.7. Then, $\mathcal{C}_z$ is well-defined and the following is true:*

(i) *For any $r \in [-\frac{1}{2}, \frac{1}{2}]$ the map $\mathcal{C}_z$ has a bounded extension or restriction (depending on r) $\mathcal{C}_z : H^r(\Sigma; \mathbb{C}^N) \to H^r(\Sigma; \mathbb{C}^N)$.*

(ii) *For any $r \in (0, \frac{1}{2}]$ and $\varphi \in H^r(\Sigma; \mathbb{C}^N)$ one has*

$$\mathcal{C}_z \varphi = \pm \frac{i}{2}(\alpha \cdot \nu)\varphi + \boldsymbol{t}_\Sigma^\pm \Phi_z \varphi.$$

Proof. (i) First, it follows from Proposition 3.6 (i) and Definition 3.7 that $\mathcal{C}_z$ is a bounded operator in $H^{1/2}(\Sigma; \mathbb{C}^N)$. Next, we show that the antidual $\mathcal{C}_{\overline{z}}'$ of $\mathcal{C}_{\overline{z}}$, which is a bounded map in $H^{-1/2}(\Sigma; \mathbb{C}^N)$, is an extension of $\mathcal{C}_z$. To see this, let $\varphi, \psi \in H^{1/2}(\Sigma; \mathbb{C}^N)$. We use (3.11), Proposition 3.6 (ii), (3.13), and the definition of $\mathcal{C}_z$ to obtain

$$\begin{aligned}
0 &= (X\widehat{\Phi}_z\varphi, \widehat{\Phi}_{\overline{z}}\psi)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} - (\widehat{\Phi}_z\varphi, X\widehat{\Phi}_{\overline{z}}\psi)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= (\mathcal{C}_z\varphi, \psi)_{L^2(\Sigma;\mathbb{C}^N)} - (\varphi, \mathcal{C}_{\overline{z}}\psi)_{L^2(\Sigma;\mathbb{C}^N)} \\
&= \langle \mathcal{C}_z\varphi, \psi\rangle_{H^{-1/2}(\Sigma;\mathbb{C}^N)\times H^{1/2}(\Sigma;\mathbb{C}^N)} - \langle \varphi, \mathcal{C}_{\overline{z}}\psi\rangle_{H^{-1/2}(\Sigma;\mathbb{C}^N)\times H^{1/2}(\Sigma;\mathbb{C}^N)} \\
&= \langle (\mathcal{C}_z - \mathcal{C}_{\overline{z}}')\varphi, \psi\rangle_{H^{-1/2}(\Sigma;\mathbb{C}^N)\times H^{1/2}(\Sigma;\mathbb{C}^N)},
\end{aligned}$$

where $\langle \cdot, \cdot\rangle_{H^{-1/2}(\Sigma;\mathbb{C}^N)\times H^{1/2}(\Sigma;\mathbb{C}^N)}$ denotes the sesquilinear duality product, which is antilinear in the second argument, on $H^{-1/2}(\Sigma; \mathbb{C}^N) \times H^{1/2}(\Sigma; \mathbb{C}^N)$. Hence, $\mathcal{C}_{\overline{z}}'$ is an extension of $\mathcal{C}_z$ which is bounded in $H^{-1/2}(\Sigma; \mathbb{C}^N)$. By interpolation, we conclude that $\mathcal{C}_z$ gives rise to a bounded map in $H^r(\Sigma; \mathbb{C}^N)$ for any $r \in [-\frac{1}{2}, \frac{1}{2}]$, which also implies that $\mathcal{C}_z$ can be extended to a bounded operator in $L^2(\Sigma; \mathbb{C}^N)$. This extension is unique since $H^{1/2}(\Sigma; \mathbb{C}^N)$ is dense in $L^2(\Sigma; \mathbb{C}^N)$, see Proposition 2.2 (ii), and hence $\mathcal{C}_z : L^2(\Sigma; \mathbb{C}^N) \to L^2(\Sigma; \mathbb{C}^N)$ is well-defined.

(ii) First, for $\varphi \in H^{1/2}(\Sigma;\mathbb{C}^N)$ Definition 3.7 and the relation (3.13) imply

$$\begin{aligned} \mathcal{C}_z\varphi &= \frac{1}{2}\boldsymbol{t}_\Sigma^+ (\Phi_z\varphi)_+ + \frac{1}{2}\boldsymbol{t}_\Sigma^- (\Phi_z\varphi)_- \\ &= \mp\frac{1}{2}\Big(\boldsymbol{t}_\Sigma^+ \big(\widehat{\Phi}_z\varphi\big)_+ - \boldsymbol{t}_\Sigma^- \big(\widehat{\Phi}_z\varphi\big)_-\Big) + \boldsymbol{t}_\Sigma^\pm(\Phi_z\varphi)_\pm \\ &= \pm\frac{i}{2}(\alpha\cdot\nu)\Gamma_0\widehat{\Phi}_z\varphi + \boldsymbol{t}_\Sigma^\pm (\Phi_z\varphi)_\pm \\ &= \pm\frac{i}{2}(\alpha\cdot\nu)\varphi + \boldsymbol{t}_\Sigma^\pm(\Phi_z\varphi)_\pm, \end{aligned} \tag{3.14}$$

which is the claimed identity for $\varphi \in H^{1/2}(\Sigma;\mathbb{C}^N)$. If $r \in (0,\frac{1}{2})$ and $\varphi \in H^r(\Sigma;\mathbb{C}^N)$, then (ii) follows from (3.14) by continuity and density; see Proposition 2.2 (ii). $\square$

3.2 Dirac operators with regular potentials

In this brief section we introduce Dirac operators with regular potentials and prove a corresponding resolvent formula.

Definition 3.9. *Let H be the free Dirac operator given by Definition 3.2 and the operator $P : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ be bounded and self-adjoint. Then, we define the self-adjoint operator H_P by*

$$H_P := H + P, \qquad \operatorname{dom} H_P = \operatorname{dom} H = H^1(\mathbb{R}^\theta;\mathbb{C}^N) \subset L^2(\mathbb{R}^\theta;\mathbb{C}^N).$$

Next, we provide a Birman-Schwinger principle and a resolvent formula for H_P. Such principles and formulas are well known in various situations in literature; see [32, 39, 45, 46]. However, for completeness we provide a short proof. We do this by proving a Birman-Schwinger principle and a resolvent formula in a general framework in Lemma 3.10. Afterwards, we apply this lemma to the operator H_P in Proposition 3.11.

Lemma 3.10. *Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert spaces, $\mathcal{T}$ be an unbounded self-adjoint operator in $\mathcal{H}$, $\mathcal{P} = \mathcal{P}_L\mathcal{P}_R$, where $\mathcal{P}_L : \mathcal{G} \to \mathcal{H}$ and $\mathcal{P}_R : \mathcal{H} \to \mathcal{G}$ are bounded operators such that $\mathcal{P} = \mathcal{P}_L\mathcal{P}_R$ is self-adjoint, $z \in \rho(\mathcal{T})$ and $\mathcal{R}(z) := (\mathcal{T} - z)^{-1}$. Then, $\mathcal{T} + \mathcal{P}$ is self-adjoint and the following is true:*

(i) $z \in \sigma_p(\mathcal{T} + \mathcal{P}) \iff -1 \in \sigma_p(\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)$.

(ii) *If $-1 \in \rho(\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)$, then $z \in \rho(\mathcal{T} + \mathcal{P})$ and*

$$(\mathcal{T} + \mathcal{P} - z)^{-1} = \mathcal{R}(z) - \mathcal{R}(z)\mathcal{P}_L(I + \mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z).$$

Proof. The operator $\mathcal{T}+\mathcal{P}$ is self-adjoint since $\mathcal{P}$ is bounded and self-adjoint. Next, we prove (i). Let $z \in \sigma_p(\mathcal{T}+\mathcal{P})$. Then, there exists a nonzero $u \in \operatorname{dom}\mathcal{T}$ such that $(\mathcal{T}-z)u+\mathcal{P}u=0$. Applying $\mathcal{R}(z)$ and using $\mathcal{P}=\mathcal{P}_L\mathcal{P}_R$ gives us the equation $u+\mathcal{R}(z)\mathcal{P}_L\mathcal{P}_R u=0$. This implies in particular $f:=\mathcal{P}_R u \neq 0$. Applying $\mathcal{P}_R$ yields $f+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L f=0$, i.e. $-1 \in \sigma_p(\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)$. Now, let $-1 \in \sigma_p(\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)$. Then, there exists a nonzero $f \in \mathcal{G}$ such that $f+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L f=0$. We define $u:=\mathcal{R}(z)\mathcal{P}_L f\neq 0$. Then,

$$(\mathcal{T}-z)u+\mathcal{P}u=\mathcal{P}_L f+\mathcal{P}_L\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L f=0,$$

implying $z\in\sigma_p(\mathcal{T}+\mathcal{P})$. To prove (ii) we assume $-1\in\rho(\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)$. Then,

$$\begin{aligned}
(\mathcal{T}+\mathcal{P}-z)&\big(\mathcal{R}(z)-\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\big)\\
&=((\mathcal{T}-z)+\mathcal{P}_L\mathcal{P}_R)\big(\mathcal{R}(z)-\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\big)\\
&=I-\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)+\mathcal{P}_L\mathcal{P}_R\mathcal{R}(z)\\
&\quad-\mathcal{P}_L\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\\
&=I-\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\\
&\quad+\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\\
&\quad-\mathcal{P}_L\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)\\
&=I.
\end{aligned}$$

Hence, $\mathcal{R}(z)-\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)$ is a right inverse of $\mathcal{T}+\mathcal{P}-z$. One shows in the same way that $\mathcal{R}(z)-\mathcal{R}(z)\mathcal{P}_L(I+\mathcal{P}_R\mathcal{R}(z)\mathcal{P}_L)^{-1}\mathcal{P}_R\mathcal{R}(z)$ is a left inverse of $\mathcal{T}+\mathcal{P}-z$. Thus, the resolvent formula for $\mathcal{T}+\mathcal{P}$ is valid. □

As a consequence of Lemma 3.10, we get the following result for the operator H_P from Definition 3.9.

Proposition 3.11. *Let $\mathcal{G}$ be a Hilbert space, $P=P_LP_R$, where $P_L:\mathcal{G}\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and $P_R:L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to\mathcal{G}$ are bounded operators such that $P=P_LP_R$ is self-adjoint, $z\in\rho(H)$ and $R(z)=(H-z)^{-1}$ be the resolvent of the free Dirac operator. Then, $H_P=H+P$ is self-adjoint and the following statements hold:*

(i) $z\in\sigma_p(H_P) \iff -1\in\sigma_p(P_RR(z)P_L)$.

(ii) *If $-1\in\rho(P_RR(z)P_L)$, then $z\in\rho(H_P)$ and*

$$(H_P-z)^{-1}=R(z)-R(z)P_L(I+P_RR(z)P_L)^{-1}P_RR(z).$$

3.3 Dirac operators with δ-shell potentials

Next, we define Dirac operators with δ-shell potentials supported on a special C^2-surface Σ. Such operators are formally given by $-i(\alpha \cdot \nabla) + m\beta + \widetilde{V}\delta_\Sigma$, where $\widetilde{V} = \widetilde{V}^* \in W^1_\infty(\Sigma; \mathbb{C}^N)$ is the interaction matrix.

Definition 3.12. *Let $m \in \mathbb{R}$, $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1 and $\widetilde{V} = \widetilde{V}^* \in W^1_\infty(\Sigma; \mathbb{C}^N)$. Then, we define the operator $H_{\widetilde{V}\delta_\Sigma}$ through*

$$\begin{aligned}\operatorname{dom} H_{\widetilde{V}\delta_\Sigma} &:= \Big\{u \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N) : i(\alpha \cdot \nu)(\boldsymbol{t}^+_\Sigma - \boldsymbol{t}^-_\Sigma)u + \frac{\widetilde{V}}{2}(\boldsymbol{t}^+_\Sigma + \boldsymbol{t}^-_\Sigma)u = 0\Big\}\\ H_{\widetilde{V}\delta_\Sigma} u &:= (-i(\alpha \cdot \nabla) + m\beta)u_+ \oplus (-i(\alpha \cdot \nabla) + m\beta)u_-.\end{aligned}$$

Using the notations introduced below Proposition 3.6 we can write the operator $H_{\widetilde{V}\delta_\Sigma}$ as $X \upharpoonright \ker(\Gamma_0 + \widetilde{V}\Gamma_1)$, where $\Gamma_0 = i(\alpha \cdot \nu)(\boldsymbol{t}^+_\Sigma - \boldsymbol{t}^-_\Sigma)$ and $\Gamma_1 = \frac{1}{2}(\boldsymbol{t}^+_\Sigma - \boldsymbol{t}^-_\Sigma)$.

The self-adjointness and spectral properties of $H_{\widetilde{V}\delta_\Sigma}$ have been investigated in numerous papers in various situations; see for instance [4, 5, 8, 9, 13, 16, 18, 19, 24, 28, 56, 59, 60] Interaction matrices having the form $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, where $\widetilde{\eta}$ and $\widetilde{\tau}$ are constants in $\mathbb{R}$ or sufficiently smooth real-valued functions on Σ, are the most prevalent in literature. They are used to model electrostatic and Lorentz scalar interactions. In this case an explicit condition for self-adjointness is known; cf. Proposition 3.15. However, in our general setting self-adjointness is not guaranteed. We start our analysis by showing that $H_{\widetilde{V}\delta_\Sigma}$ is a densely defined symmetric operator in Lemma 3.13. Afterwards, we show in Proposition 3.14 the strong connection between $H_{\widetilde{V}\delta_\Sigma}$ and $I + \mathcal{C}_z\widetilde{V}$, where $\mathcal{C}_z$ is the operator introduced in Definition 3.7.

Lemma 3.13. *Let $\widetilde{V} = \widetilde{V}^* \in W^1_\infty(\Sigma; \mathbb{C}^N)$. Then, the operator $H_{\widetilde{V}\delta_\Sigma}$ is symmetric and densely defined.*

Proof. By [54, Corollary 3.5] the set $C^\infty_0(\overline{\Omega_+}; \mathbb{C}^N) \oplus C^\infty_0(\overline{\Omega_-}; \mathbb{C}^N) \subset \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ is dense in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$. Consequently, the operator $H_{\widetilde{V}\delta_\Sigma}$ is densely defined. Furthermore, since $H_{\widetilde{V}\delta_\Sigma} = X \upharpoonright \ker(\Gamma_0 + \widetilde{V}\Gamma_1)$, we obtain with (3.11)

$$\begin{aligned}(H_{\widetilde{V}\delta_\Sigma} u, v)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} - (u, H_{\widetilde{V}\delta_\Sigma} v)_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} &= (\Gamma_1 u, \Gamma_0 v)_{L^2(\Sigma;\mathbb{C}^N)} - (\Gamma_0 u, \Gamma_1 v)_{L^2(\Sigma;\mathbb{C}^N)}\\ &= -(\Gamma_1 u, \widetilde{V}\Gamma_1 v)_{L^2(\Sigma;\mathbb{C}^N)} + (\widetilde{V}\Gamma_1 u, \Gamma_1 v)_{L^2(\Sigma;\mathbb{C}^N)}\\ &= 0 \qquad \forall u, v \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}.\end{aligned}$$

Thus, $H_{\widetilde{V}\delta_\Sigma}$ is symmetric. $\square$

Proposition 3.14. *Let $\widetilde{V} = \widetilde{V}^* \in W^1_\infty(\Sigma; \mathbb{C}^N)$, $z \in \rho(H)$, $R(z) = (H - z)^{-1}$ be the resolvent of the free Dirac operator, Φ_z be defined as in Definition 3.5 and $\mathcal{C}_z$ be defined as in Definition 3.7. Then, $H_{\widetilde{V}\delta_\Sigma}$ is closed and $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$, if and only if $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$. Moreover, in this case the resolvent formula*

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = R(z) - \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}} \tag{3.15}$$

applies.

Proof. We start by showing that if $H_{\widetilde{V}\delta_\Sigma}$ is closed and $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$, then $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$. By Proposition 2.2 (iii) and Proposition 3.8 (i) $\widetilde{V}$ and $\mathcal{C}_z$ act as bounded operators in $H^{1/2}(\Sigma; \mathbb{C}^N)$, respectively. Hence, it suffices to show that $I + \widetilde{V}\mathcal{C}_z$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$; then Proposition 2.29 shows that $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$. Moreover, since $I + \widetilde{V}\mathcal{C}_z$ is a bounded operator in $H^{1/2}(\Sigma; \mathbb{C}^N)$, we only have to prove that $I + \widetilde{V}\mathcal{C}_z$ is bijective. Let $\psi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ such that $(I + \widetilde{V}\mathcal{C}_z)\psi = 0$. We set $u = \Phi_z\psi$. Then, Proposition 3.6 (i) implies $u \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. Furthermore, Proposition 3.6 (ii) yields $(-i(\alpha \cdot \nabla) + m\beta - zI_N)u_\pm = 0$ and Proposition 3.8 (ii) gives us

$$\Gamma_0 u = \psi \quad \text{and} \quad \Gamma_1 u = \mathcal{C}_z\psi. \tag{3.16}$$

Thus, (3.16) leads to

$$\Gamma_0 u + \widetilde{V}\Gamma_1 u = \psi + \widetilde{V}\mathcal{C}_z\psi = 0$$

and hence $u \in \ker(H_{\widetilde{V}\delta_\Sigma} - z)$. We have $\ker(H_{\widetilde{V}\delta_\Sigma} - z) = \{0\}$ since $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$. This implies $u = 0$ and therefore (3.16) shows $\psi = 0$. Now, we turn to the surjectivity. Let $\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$. Then, according to Proposition 2.3 (i) there exist functions $w_\pm \in H^1(\Omega_\pm; \mathbb{C}^N)$ such that $\boldsymbol{t}^\pm_\Sigma w_\pm = \frac{\mp i(\alpha\cdot\nu)}{2}\varphi \in H^{1/2}(\Sigma; \mathbb{C}^N)$. Next, we define $w := w_+ \oplus w_- \in H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ and see

$$\Gamma_0 w = \varphi \quad \text{as well as} \quad \Gamma_1 w = 0. \tag{3.17}$$

Moreover, let

$$v := (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}(X - z)w,$$

where X is the operator introduced in (3.10). By definition, $v \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma} \subset \operatorname{dom} X$ and $(X - z)(w - v)_\pm = 0$, and thus due to (3.13) and the text below there exists a $\psi \in H^{1/2}(\Sigma; \mathbb{C}^N)$ such that $\Phi_z\psi = w - v$. Hence, we use the relations (3.16) (for $u = \Phi_z\psi$), (3.17) and $v \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ to obtain

$$\begin{aligned}(I + \widetilde{V}\mathcal{C}_z)\psi &= \Gamma_0\Phi_z\psi + \widetilde{V}\Gamma_1\Phi_z\psi \\ &= \Gamma_0(w - v) + \widetilde{V}\Gamma_1(w - v) \\ &= \Gamma_0 w + \widetilde{V}\Gamma_1 w = \varphi.\end{aligned}$$

This proves the surjectivity.

Next, let us prove the reverse direction. We assume that $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Then, by Proposition 2.29 $I + \widetilde{V}\mathcal{C}_z$ is also continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$. We start by showing that $\ker(H_{\widetilde{V}\delta_\Sigma} - z) = \{0\}$ in this case. To do so, we assume that $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ and $(H_{\widetilde{V}\delta_\Sigma} - z)u = 0$. Then, according to (3.13) and the text below there exists a $\psi \in H^{1/2}(\Sigma;\mathbb{C}^N)$ such that $u = \Phi_z\psi$. Since u satisfies the boundary conditions, the equation $\Gamma_0 u + \widetilde{V}\Gamma_1 u = 0$ is valid which implies by Proposition 3.8 (ii)

$$\Gamma_0\Phi_z\psi + \widetilde{V}\Gamma_1\Phi_z\psi = (I + \widetilde{V}\mathcal{C}_z)\psi = 0.$$

Hence, $\psi = 0$ and in turn also $u = 0$, which shows $\ker(H_{\widetilde{V}\delta_\Sigma} - z) = \{0\}$. Finally, we show that the expression $R(z) - \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^*$ is a right inverse of $H_{\widetilde{V}\delta_\Sigma} - z$. This implies $\operatorname{ran}(H_{\widetilde{V}\delta_\Sigma} - z) = L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and (3.15). Moreover, as the right-hand side of (3.15) is bounded in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ by Proposition 3.3 (iii) and Proposition 3.6, this also implies that $H_{\widetilde{V}\delta_\Sigma}$ is closed and $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$.
We start by choosing $v \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and setting

$$u := R(z)v - \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v.$$

Using $\operatorname{dom} H = H^1(\mathbb{R}^\theta;\mathbb{C}^N)$, Proposition 3.6 (iii) and Proposition 3.8 (ii) (applied to $\varphi = \widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v$) yields

$$\begin{aligned}
&\widetilde{V}\Gamma_1 u + \Gamma_0 u \\
&\quad = \widetilde{V}\Phi_{\overline{z}}^* v - \widetilde{V}\mathcal{C}_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v - i(\alpha\cdot\nu)(-i(\alpha\cdot\nu))\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v \\
&\quad = \widetilde{V}\Phi_{\overline{z}}^* v - \widetilde{V}(I + \mathcal{C}_z\widetilde{V})(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v \\
&\quad = 0.
\end{aligned}$$

Hence, $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$. We get with $(-i(\alpha\cdot\nabla) + m\beta - zI_N)R(z)v = v$ and Proposition 3.6 (ii) that

$$\begin{aligned}
\big((H_{\widetilde{V}\delta_\Sigma} - z)u\big)_\pm &= (-i(\alpha\cdot\nabla) + m\beta - zI_N)u_\pm \\
&= (-i(\alpha\cdot\nabla) + m\beta - zI_N)(R(z)v)_\pm \\
&\qquad - (-i(\alpha\cdot\nabla) + m\beta - zI_N)\big(\Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_{\overline{z}}^* v\big)_\pm \\
&= v_\pm,
\end{aligned}$$

which concludes the proof. □

The previous proposition can be used to show the self-adjointness of $H_{\widetilde{V}\delta_\Sigma}$. For instance if $I + \mathcal{C}_z\widetilde{V}$ and $I + \mathcal{C}_{\overline{z}}\widetilde{V}$ are continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$, then it follows directly by Lemma 3.13 and Proposition 3.14 that $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint.

For the rest of this section we focus on interaction matrices of the form $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ with $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$. Properties of such Dirac operators which are important with respect to this thesis are given in the following proposition.

Proposition 3.15. *Let $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ with $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$ and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$. If*

$$\inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma) - 4| > 0,$$

then $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint. Moreover, in this case the following is true:

(i) *If $\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma)| > 0$, then $UH_{\widetilde{V}\delta_\Sigma}U = H_{-4(\widetilde{V}/\widetilde{d})\delta_\Sigma}$, where U is the self-adjoint unitary operator*

$$U : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N), \qquad Uu := u_+ \oplus (-u_-).$$

(ii) *If $\widetilde{d} = -4$, then $H_{\widetilde{V}\delta_\Sigma}$ induces confinement, i.e. $H_{\widetilde{V}\delta_\Sigma} = H^+_{\widetilde{V}} \oplus H^-_{\widetilde{V}}$, where $H^\pm_{\widetilde{V}}$ are operators acting in $L^2(\Omega_\pm;\mathbb{C}^N)$ given by*

$$\begin{aligned} H^\pm_{\widetilde{V}} u_\pm &= -i(\alpha\cdot\nabla)u_\pm + m\beta u_\pm, \\ \operatorname{dom} H^\pm_{\widetilde{V}} &= \big\{ u_\pm \in H^1(\Omega_\pm;\mathbb{C}^N) : (2I_N \mp i(\alpha\cdot\nu)\widetilde{V})\boldsymbol{t}^\pm_\Sigma u_\pm = 0 \big\} \subset L^2(\Omega_\pm;\mathbb{C}^N). \end{aligned}$$

(iii) *The operator $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^r(\Sigma;\mathbb{C}^N)$ for all $r \in [0,\frac{1}{2}]$ and $z \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. The self-adjointness of $H_{\widetilde{V}\delta_\Sigma}$ follows from [60, Section 6]. Item (i) and (ii) are well known and have been shown in numerous settings, see for instance [24, Section 4], [37, Theorem 2.3 (d)] or [49] for (i), and [5, Theorem 5.5], [16, Section 5.2] or [60, Example 12] for (ii). Although the proofs do not change in our case, we provide them for completeness. Let us start with (i). After checking the definitions of U and $H_{\widetilde{V}\delta_\Sigma}$ one sees that we only have to prove for $u \in H^1(\mathbb{R}^\theta\setminus\Sigma;\mathbb{C}^N)$

$$Uu \in \ker\big(\Gamma_0 + \widetilde{V}\Gamma_1\big) \iff u \in \ker\big(\Gamma_0 + \tfrac{-4\widetilde{V}}{\widetilde{d}}\Gamma_1\big),$$

where Γ_1 and Γ_0 are the operators introduced above (3.10). Using (3.1), the identity $\widetilde{V}(\alpha\cdot\nu)\widetilde{V} = (\alpha\cdot\nu)\widetilde{d}$ and the definition of U yields

$$\begin{aligned} & & \Gamma_0 Uu + \widetilde{V}\Gamma_1 Uu = 0 \\ &\iff & i(\alpha\cdot\nu)(\boldsymbol{t}^+_\Sigma u_+ - \boldsymbol{t}^-_\Sigma(-u_-)) + \widetilde{V}\frac{1}{2}(\boldsymbol{t}^+_\Sigma u_+ + \boldsymbol{t}^-_\Sigma(-u_-)) = 0 \\ &\iff & -(\boldsymbol{t}^+_\Sigma u_+ + \boldsymbol{t}^-_\Sigma u_-) + i(\alpha\cdot\nu)\widetilde{V}\frac{1}{2}(\boldsymbol{t}^+_\Sigma u_+ - \boldsymbol{t}^-_\Sigma u_-) = 0 \\ &\iff & -4\widetilde{V}\frac{1}{2}(\boldsymbol{t}^+_\Sigma u_+ + \boldsymbol{t}^-_\Sigma u_-) + \widetilde{d}i(\alpha\cdot\nu)(\boldsymbol{t}^+_\Sigma u_+ - \boldsymbol{t}^-_\Sigma u_-) = 0 \\ &\iff & -4\widetilde{V}\Gamma_1 + \widetilde{d}\Gamma_0 u = 0 \\ &\iff & \Gamma_0 u + \frac{-4\widetilde{V}}{\widetilde{d}}\Gamma_1 u = 0, \end{aligned}$$

which proves (i).

Next, let us consider (ii). In this setting $\widetilde{d} = -4$. Moreover, it suffices to show that the domains of $H_{\widetilde{V}\delta_\Sigma}$ and $H^+_{\widetilde{V}} \oplus H^-_{\widetilde{V}}$ coincide. Let us start by assuming that $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$. It is easy to see that the transmission condition $\Gamma_0 u + \widetilde{V}\Gamma_1 u = 0$ can be rewritten in the form

$$(2I_N - i(\alpha \cdot \nu)\widetilde{V})\boldsymbol{t}^+_\Sigma u_+ = (2I_N + i(\alpha \cdot \nu)\widetilde{V})\boldsymbol{t}^-_\Sigma u_-. \tag{3.18}$$

Multiplying this equation with $(2I_N \mp i(\alpha\cdot\nu)\widetilde{V})$ and using $((\alpha\cdot\nu)\widetilde{V})^2 = \widetilde{d}I_N = -4I_N$ yields

$$4(2I_N \mp i(\alpha \cdot \nu)\widetilde{V})\boldsymbol{t}^\pm_\Sigma u_\pm = 0.$$

Hence, $u_\pm \in \operatorname{dom} H^\pm_{\widetilde{V}}$ and also $u = u_+ \oplus u_- \in \operatorname{dom} H^+_{\widetilde{V}} \oplus H^-_{\widetilde{V}}$. If $u \in \operatorname{dom} H^+_{\widetilde{V}} \oplus H^-_{\widetilde{V}}$, then (3.18) is fulfilled since both sides of the equation are zero. Thus, the equation $\Gamma_0 u + \widetilde{V}\Gamma_1 u = 0$ is also valid, and therefore $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$.

It remains to prove (iii). For $r = \frac{1}{2}$ the assertion follows from Proposition 3.14 since $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint. Next, let us consider the case $r \in [0, 1/2)$. From the proof of Proposition 3.8 we know that $(\mathcal{C}_{\overline{z}} \restriction H^{1/2}(\Sigma;\mathbb{C}^N))'$, where $'$ is used to denote the antidual operator, is a continuous extension of $\mathcal{C}_z$ to $H^{-1/2}(\Sigma;\mathbb{C}^N)$. Moreover, using the symmetry of $\widetilde{V}$ and the fact that $\widetilde{V}$ induces a bounded multiplication operator in $H^{1/2}(\Sigma;\mathbb{C}^N)$ shows that $\widetilde{V}$ can also be extended to a bounded multiplication operator in $H^{-1/2}(\Sigma;\mathbb{C}^N)$. Therefore,

$$\big((I + \widetilde{V}\mathcal{C}_{\overline{z}}) \restriction H^{1/2}(\Sigma;\mathbb{C}^N)\big)' = I + (\mathcal{C}_{\overline{z}} \restriction H^{1/2}(\Sigma;\mathbb{C}^N))'(\widetilde{V} \restriction H^{1/2}(\Sigma;\mathbb{C}^N))' \tag{3.19}$$

is a continuous extension of $I + \mathcal{C}_z\widetilde{V}$ to $H^{-1/2}(\Sigma;\mathbb{C}^N)$. Since $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint, we can apply Proposition 3.14 again and obtain that $I + \mathcal{C}_{\overline{z}}\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Thus, according to Proposition 2.29 the operator $I + \widetilde{V}\mathcal{C}_{\overline{z}}$ is also continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Therefore, the operator in (3.19) has the bounded inverse $\big((I + \widetilde{V}\mathcal{C}_{\overline{z}})^{-1} \restriction H^{1/2}(\Sigma;\mathbb{C}^N)\big)'$. Hence, one can use interpolation to show the assertion for $r \in [0, \frac{1}{2})$; cf. Section 2.1 (xiii) and Proposition 2.2 (i). $\square$

4 Norm resolvent convergence of Dirac operators with general strongly localized potentials

In this chapter, which is based on [14, Section 3], we find in Theorem 4.15 conditions for the norm resolvent convergence of Dirac operators with strongly localized potentials. Moreover, we show in Corollary 4.16 that these conditions are fulfilled if the potential satisfies a certain smallness condition.

Throughout this chapter we assume that $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$, $\theta \in \{2,3\}$, is a special C^2-surface as in Definition 2.1 and $\varepsilon_{\mathrm{tub}} \in (0,\infty)$ is chosen as in Proposition 2.12. Moreover, we assume that

$$q \in L^\infty((-1,1);\mathbb{R}) \quad \text{with} \quad \int_{-1}^{1} q(t)\,dt = 1 \tag{4.1}$$

and

$$V = V^* \in W^1_\infty(\Sigma;\mathbb{C}^N), \tag{4.2}$$

which we call the *interaction matrix*. For $\varepsilon \in (0,\varepsilon_{\mathrm{tub}})$ we define

$$V_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} V(x_\Sigma) q\big(\frac{t}{\varepsilon}\big), & x = \iota(x_\Sigma,t) \in \Omega_\varepsilon, \\ 0, & x \notin \Omega_\varepsilon, \end{cases} \tag{4.3}$$

where $\iota(x_\Sigma,t) = x_\Sigma + t\nu(x_\Sigma)$ for $(x_\Sigma,t) \in \Sigma \times \mathbb{R}$; cf. Definition 2.7. By Proposition 2.12 the strongly localized potential V_ε is well-defined. Furthermore, the properties of q and V imply $V_\varepsilon = V_\varepsilon^* \in L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$. In this chapter we study the norm resolvent convergence properties of the self-adjoint operators $H_{V_\varepsilon} = H + V_\varepsilon$, $\varepsilon \in (0,\varepsilon_{\mathrm{tub}})$, which are explicitly given by

$$H_{V_\varepsilon} = -i(\alpha\cdot\nabla) + m\beta + V_\varepsilon, \qquad \operatorname{dom} H_{V_\varepsilon} = H^1(\mathbb{R}^\theta;\mathbb{C}^N). \tag{4.4}$$

We do this by finding a suitable resolvent formula in terms of Bochner-integral operators, denoted by $A_\varepsilon(z), B_\varepsilon(z)$ and $C_\varepsilon(z)$, in Section 4.1. Afterwards we introduce a shift operator which allows us to prove convergence properties of $A_\varepsilon(z), B_\varepsilon(z)$ and $C_\varepsilon(z)$ in Section 4.3. In Section 4.4 we study the limits of these operators and connect them to the resolvent of a Dirac operator with a δ-shell potential supported on Σ and a rescaled interaction matrix $\widetilde{V} = \widetilde{V}^* \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. Lastly, we find in Section 4.5 a condition for the norm resolvent convergence of H_{V_ε} as $\varepsilon \to 0$.

4.1 A resolvent formula for Dirac operators with strongly localized potentials

In this section we find a suitable resolvent formula for H_{V_ε}. In order to be able to apply Proposition 3.11, we need an appropriate factorisation for the potential V_ε. We recall that Ω_ε was defined in Definition 2.7 and introduce the mappings

$$\begin{aligned} &\mathcal{I}_\varepsilon : L^2((-\varepsilon,\varepsilon); L^2(\Sigma;\mathbb{C}^N)) \to L^2(\Omega_\varepsilon;\mathbb{C}^N), \quad \mathcal{I}_\varepsilon f(\iota(x_\Sigma,t)) := f(t)(x_\Sigma),\\ &\mathcal{I}_\varepsilon^{-1} : L^2(\Omega_\varepsilon;\mathbb{C}^N) \to L^2((-\varepsilon,\varepsilon); L^2(\Sigma;\mathbb{C}^N)), \quad \mathcal{I}_\varepsilon^{-1} u(t)(x_\Sigma) := u(\iota(x_\Sigma,t)), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\mathcal{S}_\varepsilon : \mathcal{B}^0(\Sigma) \to L^2((-\varepsilon,\varepsilon); L^2(\Sigma;\mathbb{C}^N)), \quad \mathcal{S}_\varepsilon g(t) := \frac{1}{\sqrt{\varepsilon}} g\big(\tfrac{t}{\varepsilon}\big),\\ &\mathcal{S}_\varepsilon^{-1} : L^2((-\varepsilon,\varepsilon); L^2(\Sigma;\mathbb{C}^N)) \to \mathcal{B}^0(\Sigma), \quad \mathcal{S}_\varepsilon^{-1} g(t) := \sqrt{\varepsilon} g(\varepsilon t), \end{aligned} \tag{4.6}$$

where $\mathcal{B}^0(\Sigma) = L^2((-1,1); L^2(\Sigma;\mathbb{C}^N))$; cf. Section 2.1 (xi). According to Proposition 2.12 and Proposition 2.18 (iii) for any $\varepsilon \in (0,\varepsilon_{\text{tub}})$ these mappings are well-defined, bounded, invertible, and their inverses have the claimed form; cf. [7, equations (3.6) and (3.7)]. Moreover, we set $u_\varepsilon := \frac{\chi_{\Omega_\varepsilon}}{\sqrt{\varepsilon}}$, where $\chi_{\Omega_\varepsilon}$ is the characteristic function for Ω_ε, and define the operators

$$U_\varepsilon : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\Omega_\varepsilon;\mathbb{C}^N) \quad \text{and} \quad U_\varepsilon^* : L^2(\Omega_\varepsilon;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N) \tag{4.7}$$

acting on $u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and $v \in L^2(\Omega_\varepsilon;\mathbb{C}^N)$ as

$$U_\varepsilon u = (u_\varepsilon u) \restriction \Omega_\varepsilon \quad \text{and} \quad U_\varepsilon^* v = \begin{cases} u_\varepsilon v & \text{in } \Omega_\varepsilon,\\ 0 & \text{in } \mathbb{R}^\theta \setminus \Omega_\varepsilon. \end{cases}$$

Using these newly introduced operators shows that when viewed as a multiplication operator V_ε can be factorised in the following way:

$$V_\varepsilon = U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon V q \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon.$$

This factorisation, defining for $z \in \rho(H)$ and $\varepsilon \in (0,\varepsilon_{\text{tub}})$ the bounded operators

$$\begin{aligned} A_\varepsilon(z) &:= R(z) U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N),\\ B_\varepsilon(z) &:= \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(z) U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma),\\ C_\varepsilon(z) &:= \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(z) : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to \mathcal{B}^0(\Sigma), \end{aligned} \tag{4.8}$$

where we used $R(z) = (H-z)^{-1}$, and applying Proposition 3.11 for $P_L = U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon V q$ $P_R = \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon$ and $P = P_L P_R = V_\varepsilon$ gives us:

Proposition 4.1. *Let q, V and V_ε, $\varepsilon \in (0,\varepsilon_{\text{tub}})$, be as in (4.1)–(4.3), $z \in \rho(H)$ and $R(z) = (H-z)^{-1}$, where H is the free Dirac operator introduced in Definition 3.2. Then, H_{V_ε} is self-adjoint and the following holds:*

(i) $z \in \sigma_p(H_{V_\varepsilon}) \iff -1 \in \sigma_p(B_\varepsilon(z)Vq)$.

(ii) *If* $-1 \in \rho(B_\varepsilon(z)Vq)$, *then* $z \in \rho(H_{V_\varepsilon})$ *and the resolvent formula*

$$(H_{V_\varepsilon} - z)^{-1} = R(z) - A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}C_\varepsilon(z)$$

holds.

Having established a resolvent formula for H_{V_ε} in terms of the operators $A_\varepsilon(z)$, $B_\varepsilon(z)$ and $C_\varepsilon(z)$, we find in Proposition 4.2 integral representations of these operators, which are more convenient than (4.8).

Proposition 4.2. *Let* $z \in \rho(H)$, G_z *be the integral kernel of* $R(z) = (H - z)^{-1}$ *given by* (3.3)–(3.4) *and* W *be the Weingarten map defined in Definition 2.11. For* $\varepsilon \in (0, \varepsilon_{\text{tub}})$ *the operators* $A_\varepsilon(z)$, $B_\varepsilon(z)$, *and* $C_\varepsilon(z)$ *defined by* (4.8) *have the integral representations*

$$\begin{aligned}
A_\varepsilon(z)f(x) &= \int_{-1}^{1}\int_\Sigma G_z(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))f(s)(y_\Sigma) \\
&\qquad\qquad\qquad\qquad \cdot \det(I - \varepsilon sW(y_\Sigma))\, d\sigma(y_\Sigma)\, ds, \\
B_\varepsilon(z)f(t)(x_\Sigma) &= \int_{-1}^{1}\int_\Sigma G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma))f(s)(y_\Sigma) \\
&\qquad\qquad\qquad\qquad \cdot \det(I - \varepsilon sW(y_\Sigma))\, d\sigma(y_\Sigma)\, ds, \\
C_\varepsilon(z)u(t)(x_\Sigma) &= \int_{\mathbb{R}^\theta} G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y)u(y)\, dy,
\end{aligned}$$

for $f \in \mathcal{B}^0(\Sigma)$, $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, *a.e.* $x \in \mathbb{R}^\theta$, *a.e.* $t \in (-1, 1)$ *and* σ*-a.e.* $x_\Sigma \in \Sigma$.

Proof. First, we prove the claim for $C_\varepsilon(z)$. Using (3.2), (4.5) and (4.6) gives us for $v \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, a.e. $t \in (-1, 1)$, and σ-a.e. $x_\Sigma \in \Sigma$

$$\begin{aligned}
C_\varepsilon(z)v(t)(x_\Sigma) &= \big(\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon R(z)v\big)(t)(x_\Sigma) \\
&= \sqrt{\varepsilon}(U_\varepsilon R(z)v)(x_\Sigma + \varepsilon t\nu(x_\Sigma)) \\
&= \int_{\mathbb{R}^\theta} G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y)v(y)\, dy.
\end{aligned}$$

Next, to show the claim for $A_\varepsilon(z)$ we choose $f \in \mathcal{B}^0(\Sigma)$. Applying (3.2), Proposition 2.12 (iii), (4.5) and (4.6) yields

$$\begin{aligned}
A_\varepsilon(z)f(x) &= \left(R(z)U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon f\right)(x)\\
&= \int_{\mathbb{R}^\theta} G_z(x-y)\left(U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon f\right)(y)\,dy\\
&= \int_{\Omega_\varepsilon} G_z(x-y)u_\varepsilon(y)\left(\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon f\right)(y)\,dy\\
&= \int_{-\varepsilon}^{\varepsilon}\int_\Sigma G_z(x-(y_\Sigma+s\nu(y_\Sigma)))\frac{1}{\sqrt{\varepsilon}}\left(\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon f\right)(y_\Sigma+s\nu(y_\Sigma))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\det(I-sW(y_\Sigma))\,d\sigma(y_\Sigma)\,ds\\
&= \int_{-\varepsilon}^{\varepsilon}\int_\Sigma G_z(x-y_\Sigma-s\nu(y_\Sigma))\frac{1}{\sqrt{\varepsilon}}(\mathcal{S}_\varepsilon f)(s)(y_\Sigma)\det(I-sW(y_\Sigma))\,d\sigma(y_\Sigma)\,ds\\
&= \int_{-\varepsilon}^{\varepsilon}\int_\Sigma G_z(x-y_\Sigma-s\nu(y_\Sigma))\frac{1}{\varepsilon}f\left(\tfrac{s}{\varepsilon}\right)(y_\Sigma)\det(I-sW(y_\Sigma))\,d\sigma(y_\Sigma)\,ds\\
&= \int_{-1}^{1}\int_\Sigma G_z(x-y_\Sigma-\varepsilon s\nu(y_\Sigma))f(s)(y_\Sigma)\det(I-\varepsilon sW(y_\Sigma))\,d\sigma(y_\Sigma)\,ds
\end{aligned}$$

for a.e. $x \in \mathbb{R}^\theta$, which is the claimed identity. The representation for $B_\varepsilon(z)$ follows by combining the last two calculations. □

4.2 The shift operator

We introduce and study a shift operator which turns out to be useful in the convergence analysis of the maps $A_\varepsilon(z), B_\varepsilon(z)$ and $C_\varepsilon(z)$ in (4.8). For this, we first provide a lemma which later allows us to construct a suitable extension of the normal vector field ν on Σ. We mention that this section can be found in a similar form in [14, Section 3.2].

Lemma 4.3. *Let $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1, $\varepsilon_{\text{tub}} > 0$ be as in Proposition 2.12 and $o \in C_b^1(\Sigma;\mathbb{R}^{n\times k})$ for $n,k \in \mathbb{N}$. Then o has an extension $\widehat{o} \in C_b^1(\mathbb{R}^\theta;\mathbb{R}^{n\times k})$ such that $\operatorname{supp}\widehat{o} \subset \Omega_{\varepsilon_{\text{tub}}}$ and $\widehat{o}(x_\Sigma + t\nu(x_\Sigma)) = o(x_\Sigma)$ for $(x_\Sigma, t) \in \Sigma \times (-\frac{\varepsilon_{\text{tub}}}{2}, \frac{\varepsilon_{\text{tub}}}{2})$.*

Proof. Before we start, recall the maps ι, ι_l, $\varkappa_l$, ν and ν_l introduced in Definition 2.7. Next, let us choose a function $\varpi \in C^1(\mathbb{R})$ such that the support of ϖ is contained in $(-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}})$, $0 \leq \varpi \leq 1$ and $\varpi = 1$ on $(-\frac{\varepsilon_{\text{tub}}}{2}, \frac{\varepsilon_{\text{tub}}}{2})$. Then, we define $\widehat{o}$ through

$$\widehat{o}(x) = \begin{cases} o(x_\Sigma)\varpi(t), & x = \iota(x_\Sigma, t) \in \Omega_{\varepsilon_{\text{tub}}} \text{ with } (x_\Sigma, t) \in \Sigma \times (-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}}),\\ 0, & x \notin \Omega_{\varepsilon_{\text{tub}}}. \end{cases} \tag{4.10}$$

By Proposition 2.12 $\widehat{o}$ is well-defined. Moreover, it is obvious that $\widehat{o}$ has all the claimed properties besides the C^1-smoothness and the boundedness of the derivatives. Hence, it remains to show $\widehat{o} \in C_b^1(\mathbb{R}^\theta; \mathbb{R}^{n\times k})$. Since $\widehat{o}$ is supported in $\Omega_{\varepsilon_{\mathrm{tub}}}$, it suffices consider $\widehat{o}$ only on $\Omega_{\varepsilon_{\mathrm{tub}}}$. To do so, we fix an $x = x_\Sigma + t\nu(x_\Sigma) = \iota(x_\Sigma, t) \in \Omega_{\varepsilon_{\mathrm{tub}}}$ with $(x_\Sigma, t) \in \Sigma \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$. According to Definition 2.1 there exists an $l \in \{1, \ldots p\}$ such that $x_\Sigma \in W_l$. Hence, we can find an $x' \in \mathbb{R}^{\theta-1}$ such that $x_\Sigma = \varkappa_l(x')$ and therefore $x = \iota_l(x', t)$. Moreover, since W_l is open, we can choose a $\delta_0 > 0$ such that $B(x_\Sigma, \delta_0) \subset W_l$. Thus, the Lipschitz continuity of $\varkappa_l$, cf. Proposition 2.9, guarantees the existence of a $\delta_1 > 0$ such that

$$\varkappa_l(B(x', \delta_1)) \subset B(x_\Sigma, \delta_0) \cap \Sigma_l \subset W_l \cap \Sigma_l \subset \Sigma,$$

implying $\iota_l(B(x', \delta_1) \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})) \subset \Omega_{\varepsilon_{\mathrm{tub}}}$ and

$$(\widehat{o} \circ \iota_l)(y', s) = o(\varkappa_l(y'))\varpi(s), \qquad \forall (y', s) \in B(x', \delta_1) \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}}). \tag{4.11}$$

Equation (4.11), $o \in C_b^1(\Sigma; \mathbb{R}^{n\times k})$ and $\varpi \in C^1(\mathbb{R}; \mathbb{R})$ show that the function

$$\widehat{o} \circ \iota_l \restriction B(x', \delta_1) \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$$

is C^1-smooth. The Jacobian of ι_l at the point $(y', s) \in B(x', \delta_1) \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$ is given by

$$\begin{aligned} &(D\iota_l)(y', s) = \begin{pmatrix}(I_\theta - sW(\varkappa_l(y'))T_l(y') & \nu_l(y')\end{pmatrix} \\ &\text{with } T_l(y') = \begin{pmatrix}(\partial_1 \varkappa_l)(y') & \ldots & (\partial_{\theta-1}\varkappa_l)(y')\end{pmatrix}, \end{aligned} \tag{4.12}$$

where $W(\varkappa_l(y'))$, which is the Weingarten map introduced in Definition 2.11, is applied column-wise to $T_l(y')$. Using the coordinate representation $L_l(y')$ of $W(\varkappa_l(y'))$ with respect to the basis $\{\partial_j \varkappa_l(x') : j = 1, \ldots, \theta - 1\}$ we obtain

$$(D\iota_l)(y', s) = \begin{pmatrix}T_l(y')(I_{\theta-1} - sL_l(y')) & \nu_l(y')\end{pmatrix}$$

and

$$\begin{aligned} ((D\iota_l)(y', s))^T (D\iota_l)(y', s) &= \begin{pmatrix}(I_{\theta-1} - s(L_l(y'))^T)(T_l(y'))^T \\ (\nu_l(y'))^T\end{pmatrix} \\ &\qquad \cdot \begin{pmatrix}T_l(y')(I_{\theta-1} - sL_l(y')) & \nu_l(y')\end{pmatrix} \\ &= \begin{pmatrix}(I_{\theta-1} - s(L_l(y'))^T)(T_l(y'))^T T_l(y')(I_{\theta-1} - sL_l(y')) & 0 \\ 0 & 1\end{pmatrix}. \end{aligned}$$

In particular, $\det((D\iota_l)(y', s))^2 = \det(I_{\theta-1} - sL_l(y'))^2 \det((T_l(y'))^T T_l(y'))$. Moreover, note that $(T_l(y'))^T T_l(y') = M_l(y') = I_{\theta-1} + \nabla\zeta_l(x')\nabla\zeta_l(y')^T$ with M_l as in the proof of Proposition 2.12, and hence $\det((T_l(y'))^T T_l(y')) = 1 + |\nabla\zeta_l(y')|^2$. If we combine this with Proposition 2.12 (ii), then we obtain

$$\begin{aligned} |\det(D\iota_l)(y', s)| &= \sqrt{1 + |\nabla\zeta_l(y')|^2}\, \det(I_{\theta-1} - sL_l(y')) \\ &= \sqrt{1 + |\nabla\zeta_l(y')|^2}\, \det(I - sW(\varkappa_l(y'))) \\ &> \frac{1}{2}. \end{aligned}$$

In particular, $(D\iota_l)(y', s)$ is invertible and thus the inverse function theorem shows that ι_l is locally around (x', t) a diffeomorphism and therefore $\widehat{o} = (\widehat{o} \circ \iota_l) \circ \iota_l^{-1}$ is C^1-smooth around the point $x = \iota_l(x', t)$. This shows that $\widehat{o}$ is C^1-smooth in $\Omega_{\varepsilon_{\mathrm{tub}}}$. Furthermore, since $|\det(D\iota_l)(y', s)| > \frac{1}{2}$ and as $(D\iota_l)(y's)$ can be bounded by a constant which only depends on ζ_l and $\varepsilon_{\mathrm{tub}}$, $((D\iota_l)(y', s))^{-1}$ can also be bounded by a constant which depends only on ζ_l and $\varepsilon_{\mathrm{tub}}$. Hence, by the chain rule and $o \in C_b^1(\Sigma; \mathbb{C}^{n\times k})$ the first order derivatives of $\widehat{o}$ are also bounded, which implies $\widehat{o} \in C_b^1(\mathbb{R}^\theta; \mathbb{C}^{n\times k})$. $\square$

Lemma 4.3 shows that the unit normal vector field ν of Σ has an extension in $C_b^1(\mathbb{R}^\theta; \mathbb{R}^\theta)$. We fix such an extension and denote it also by ν. Next, we define for $\delta \in \mathbb{R}$ the *shift operator*

$$\begin{aligned} \tau_\delta : L^2(\mathbb{R}^\theta; \mathbb{C}^N) &\to L^2(\mathbb{R}^\theta; \mathbb{C}^N) \\ \tau_\delta u(x) &:= u\left(x + \delta\nu(x)\right), \quad x \in \mathbb{R}^\theta. \end{aligned} \tag{4.13}$$

In the upcoming proposition we study basic properties of τ_δ.

Proposition 4.4. *Let $D\nu$ be the Jacobian matrix (of the extension) of ν, $r \in [0, 1]$ and $\delta_0 \in (0, \|D\nu\|^{-1}_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})$. Then, the operators τ_δ, $\delta \in [-\delta_0, \delta_0]$, are uniformly bounded in $H^r(\mathbb{R}^\theta; \mathbb{C}^N)$ and for $r' \in [0, r]$ the inequality*

$$\|\tau_\delta - I\|_{H^r(\mathbb{R}^\theta;\mathbb{C}^N)\to H^{r'}(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C|\delta|^{r-r'} \tag{4.14}$$

holds for all $\delta \in [-\delta_0, \delta_0]$, where $C > 0$ is independent of δ.

Proof. Fix $\delta \in [-\delta_0, \delta_0]$ and observe first that $I_\theta + \delta D\nu(x)$ is invertible for all $x \in \mathbb{R}^\theta$ and the norm of the inverse is bounded by $(1 - |\delta_0|\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})^{-1}$. The same bound holds for the modulus of the eigenvalues of $(I_\theta + \delta D\nu(x))^{-1}$ and hence we conclude

$$\left|\det\left((I_\theta + \delta D\nu(x))^{-1}\right)\right| \leq \frac{1}{\left(1 - |\delta_0|\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})}\right)^\theta}, \quad x \in \mathbb{R}^\theta. \tag{4.15}$$

We start by showing the uniform boundedness of τ_δ for $r = 0$. Let $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$. Then, a change of variables and (4.15) lead to

$$\begin{aligned} \int_{\mathbb{R}^\theta} |\tau_\delta u(x)|^2\, dx &= \int_{\mathbb{R}^\theta} |u(x + \delta\nu(x))|^2\, dx \\ &= \int_{\mathbb{R}^\theta} |u(x + \delta\nu(x))|^2 \frac{|\det(I_\theta + \delta D\nu(x))|}{|\det\left(I_\theta + \delta D\nu(x)\right)|}\, dx \\ &= \int_{\mathbb{R}^\theta} |u(x + \delta\nu(x))|^2 |\det(I_\theta + \delta D\nu(x))| \left|\det\left((I_\theta + \delta D\nu(x))^{-1}\right)\right| dx \\ &\leq \frac{1}{(1 - |\delta_0|\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})^\theta} \int_{\mathbb{R}^\theta} |u(x)|^2\, dx, \end{aligned} \tag{4.16}$$

and it follows that the operators τ_δ, $\delta \in [-\delta_0, \delta_0]$, are uniformly bounded in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$. To see the uniform boundedness of the operators τ_δ in $H^1(\mathbb{R}^\theta; \mathbb{C}^N)$, let $u \in C_0^\infty(\mathbb{R}^\theta; \mathbb{C}^N)$ and compute in a similar way as above

$$\begin{aligned}\int_{\mathbb{R}^\theta} |D(\tau_\delta u)(x)|^2\,dx &= \int_{\mathbb{R}^\theta} |(Du)(x+\delta\nu(x))(I_\theta + \delta D\nu(x))|^2\,dx \\ &\le \frac{\big(1+\delta_0\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})}\big)^2}{\big(1-|\delta_0|\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})}\big)^\theta} \int_{\mathbb{R}^\theta} |Du(x)|^2\,dx.\end{aligned} \tag{4.17}$$

By density this estimate remains valid for $u \in H^1(\mathbb{R}^\theta; \mathbb{C}^N)$. Therefore, the uniform boundedness of the operators τ_δ in $H^1(\mathbb{R}^\theta; \mathbb{C}^N)$ follows from (4.16) and (4.17). Eventually, using interpolation one concludes that τ_δ is uniformly bounded in $H^r(\mathbb{R}^\theta; \mathbb{C}^N)$ for any $r \in [0,1]$.

It remains to prove (4.14). Since we have already shown that τ_δ is uniformly bounded in $H^r(\mathbb{R}^\theta; \mathbb{C}^N)$, the claim in (4.14) holds for $r = r' \in [0,1]$. Next, we show (4.14) for $r' = 0$ and $r = 1$. With the main theorem of calculus and the chain rule we find for $u \in C_0^\infty(\mathbb{R}^\theta; \mathbb{C}^N)$

$$\begin{aligned}\int_{\mathbb{R}^\theta} |\tau_\delta u(x) - u(x)|^2\,dx &= \int_{\mathbb{R}^\theta} \Big|\int_0^\delta \frac{d}{dt} u(x+t\nu(x))dt\Big|^2\,dx \\ &= \int_{\mathbb{R}^\theta} \Big|\int_0^\delta Du(x+t\nu(x))\nu(x)dt\Big|^2\,dx \\ &\le \int_{\mathbb{R}^\theta} \Big(\int_0^\delta |(\tau_t Du)(x)|^2\,dt\Big)\Big(\int_0^\delta |\nu(x)|^2\,dt\Big)\,dx \\ &\le |\delta|\|\nu\|^2_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^\theta)} \int_0^\delta \|\tau_t Du\|^2_{L^2(\mathbb{R}^\theta;\mathbb{C}^{N\times\theta})}\,dt \\ &\le C|\delta| \int_0^\delta \|Du\|^2_{L^2(\mathbb{R}^\theta;\mathbb{C}^{N\times\theta})}\,dt \\ &\le C|\delta|^2 \|u\|^2_{H^1(\mathbb{R}^\theta;\mathbb{C}^N)},\end{aligned}$$

where $\tau_t Du$ is understood column-wise. By density this estimate is also valid for $u \in H^1(\mathbb{R}^\theta; \mathbb{C}^N)$ and hence $\|\tau_\delta - I\|_{H^1(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C|\delta|$. It remains to prove the claim in the case $0 \le r' < r \le 1$ with $(r', r) \ne (0,1)$. We set $\mu = r - r' \in (0,1)$ and $\upsilon = \frac{r'}{1-(r-r')} \in [0,1]$. Then,

$$r' = (1-\mu)\upsilon + \mu 0 \quad \text{and} \quad r = (1-\mu)\upsilon + \mu 1$$

and consequently [54, Theorem B.7] implies

$$H^{r'}(\mathbb{R}^\theta; \mathbb{C}^N) = \big[H^\upsilon(\mathbb{R}^\theta; \mathbb{C}^N), H^0(\mathbb{R}^\theta; \mathbb{C}^N)\big]_\mu = \big[H^\upsilon(\mathbb{R}^\theta; \mathbb{C}^N), L^2(\mathbb{R}^\theta; \mathbb{C}^N)\big]_\mu$$

and
$$H^r(\mathbb{R}^\theta;\mathbb{C}^N) = \big[H^\upsilon(\mathbb{R}^\theta;\mathbb{C}^N), H^1(\mathbb{R}^\theta;\mathbb{C}^N)\big]_\mu.$$
Applying (xiii) from Section 2.1 yields
$$\begin{aligned}
&\|I-\tau_\delta\|_{H^r(\mathbb{R}^\theta;\mathbb{C}^N)\to H^{r'}(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\le C\|I-\tau_\delta\|_{[H^\upsilon(\mathbb{R}^\theta;\mathbb{C}^N),H^1(\mathbb{R}^\theta;\mathbb{C}^N)]_\mu\to[H^\upsilon(\mathbb{R}^\theta;\mathbb{C}^N),L^2(\mathbb{R}^\theta;\mathbb{C}^N)]_\mu}\\
&\quad\le C\|I-\tau_\delta\|^{1-\mu}_{H^\upsilon(\mathbb{R}^\theta;\mathbb{C}^N)\to H^\upsilon(\mathbb{R}^\theta;\mathbb{C}^N)}\|I-\tau_\delta\|^{\mu}_{H^1(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad= C|\delta|^{r-r'},
\end{aligned}$$
which is exactly (4.14). This finishes the proof of this proposition. □

We will also need a variant of the shift operator τ_δ that acts on functions defined on $\Omega_\pm$. Since $\Sigma = \partial\Omega_\pm$ fulfils Definition 2.1, we can make use of Stein's extension operator $E : L^2(\Omega_\pm;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ given by Proposition 2.6. We then define the shift operator for functions on $\Omega_\pm$ by
$$\tau_\delta^{\Omega_\pm} := (\tau_\delta E(\cdot))_\pm : L^2(\Omega_\pm;\mathbb{C}^N) \to L^2(\Omega_\pm;\mathbb{C}^N). \tag{4.18}$$
The following properties of $\tau_\delta^{\Omega_\pm}$ immediately follow from the properties of E and Proposition 4.4.

Corollary 4.5. *Let $D\nu$ be the Jacobian matrix (of the extension) of ν, $r \in [0,1]$ and $\delta_0 \in (0, \|D\nu\|^{-1}_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})$. Then, the operators $\tau_\delta^{\Omega_\pm}$, $\delta \in [-\delta_0,\delta_0]$, are uniformly bounded in $H^r(\Omega_\pm;\mathbb{C}^N)$ and for $r' \in [0,r]$ the inequality*
$$\|\tau_\delta^{\Omega_\pm} - I\|_{H^r(\Omega_\pm;\mathbb{C}^N)\to H^{r'}(\Omega_\pm;\mathbb{C}^N)} \le C|\delta|^{r-r'}$$
holds for all $\delta \in [-\delta_0,\delta_0]$, where $C > 0$ is independent of δ.

Finally, we show that the map $t \mapsto \tau_{t\delta}u$ has a useful continuity property.

Proposition 4.6. *Let $D\nu$ be the Jacobian matrix (of the extension) of ν, $r \in [0,1]$, $\delta_0 \in (0, \|D\nu\|^{-1}_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})})$, $\delta \in [-\delta_0,\delta_0]$, $u \in H^r(\mathbb{R}^\theta;\mathbb{C}^N)$ and $v \in H^r(\Omega_\pm;\mathbb{C}^N)$. Then, the functions*
$$f_u : (-1,1) \to H^r(\mathbb{R}^\theta;\mathbb{C}^N), \qquad t \mapsto \tau_{t\delta}u,$$
and
$$f_v^\pm : (-1,1) \to H^r(\Omega_\pm;\mathbb{C}^N), \qquad t \mapsto \tau_{t\delta}^{\Omega_\pm}v,$$
are continuous.

Proof. First, consider $u \in C_0^\infty(\mathbb{R}^\theta;\mathbb{C}^N)$ and let $t_n, t \in (-1,1)$ such that $t_n \to t$ as $n \to \infty$. Then, with dominated convergence one gets

$$\lim_{n\to\infty} f_u(t_n) = \lim_{n\to\infty} u((\cdot) + \delta t_n \nu) = u((\cdot)+\delta t \nu) = f_u(t) \quad \text{in } H^1(\mathbb{R}^\theta;\mathbb{C}^N).$$

Since $H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ is continuously embedded in $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$, the assertion follows for $u \in C_0^\infty(\mathbb{R}^\theta;\mathbb{C}^N)$. If $u \in H^r(\mathbb{R}^\theta;\mathbb{C}^N)$, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^\theta;\mathbb{C}^N)$ such that $u_n \to u$ in $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$ as $n\to\infty$. Applying Proposition 4.4 yields

$$\|f_u(t) - f_{u_n}(t)\|_{H^r(\mathbb{R}^\theta;\mathbb{C}^N)} = \|\tau_{\delta t}(u-u_n)\|_{H^r(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\|u-u_n\|_{H^r(\mathbb{R}^\theta;\mathbb{C}^N)}$$

for all $n \in \mathbb{N}$ and $t \in (-1,1)$. Hence, $f_{u_n}(t) \to f_u(t)$ uniformly with respect to $t \in (-1,1)$ in $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$ as $n \to \infty$. Thus, f_u is also continuous.

It remains to verify the claim for $f_v^\pm$. Let $t_n, t \in (-1,1)$ such that $t_n \to t$ for $n\to\infty$. Using the properties of Stein's extension operator E and the above observations, we get that $f_{Ev}(t_n) \to f_{Ev}(t)$ in $H^r(\mathbb{R}^\theta;\mathbb{C}^N)$. Moreover, the boundedness of the restriction mapping gives us that $f_v^\pm(t_n) = (\tau_{\delta t_n} Ev)_\pm = (f_{Ev}(t_n))_\pm$ converges to $(f_{Ev}(t))_\pm = f_v^\pm(t)$ in $H^r(\Omega_\pm;\mathbb{C}^N)$. This shows the continuity of $f_v^\pm$. □

4.3 Convergence of $A_\varepsilon(z)$, $B_\varepsilon(z)$ and $C_\varepsilon(z)$

This section is devoted to the convergence analysis of the operators $A_\varepsilon(z), B_\varepsilon(z)$ and $C_\varepsilon(z)$ introduced in (4.8) for $\varepsilon \to 0$, and is based on [14, Section 3.3]. First, in Proposition 4.8 we study the convergence of $C_\varepsilon(z)$. Then, a duality argument allows us to investigate the convergence of $A_\varepsilon(z)$ in Proposition 4.9. Eventually, in Proposition 4.10 we consider the convergence of $B_\varepsilon(z)$.

We choose the number $\varepsilon_{ABC} > 0$ such that

$$\varepsilon_{ABC} \leq \min\left\{\frac{\varepsilon_{\text{tub}}}{4}, \frac{1}{2\|D\nu\|_{L^\infty(\mathbb{R}^\theta;\mathbb{R}^{\theta\times\theta})}}\right\}, \tag{4.19}$$

where ε_{tub} is specified in Proposition 2.12. Let W be the Weingarten map associated with Σ introduced in Definition 2.11. In our analysis, the multiplication operator $M_\varepsilon : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma)$ acting as

$$M_\varepsilon f(t) = \det(I - t\varepsilon W)\, f(t) \qquad \text{for a.e. } t \in (-1,1) \tag{4.20}$$

turns out to be useful. In the following lemma, which is an immediate consequence of Proposition 2.12 (ii) and Proposition 2.19, some relevant properties of M_ε are stated.

Lemma 4.7. *For any $\varepsilon \in (0, \varepsilon_{ABC})$ the operator M_ε is bounded, invertible,*

$$\|M_\varepsilon\|_{0\to 0} \leq (1 + \varepsilon C) \quad \text{and} \quad \|M_\varepsilon - I\|_{0\to 0} \leq \varepsilon C.$$

To formulate the result concerning the convergence of $C_\varepsilon(z)$, we recall that the embedding $\mathfrak{J}$ is defined by (2.10) and introduce the operator

$$C_0(z) := \mathfrak{J}\Phi^*_{\overline{z}} : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to \mathcal{B}^0(\Sigma). \tag{4.21}$$

In fact, the properties of $\mathfrak{J}$ and $\Phi^*_{\overline{z}}$, see (2.10) and Proposition 3.6 (iii), imply that $C_0(z)$ gives also rise to a bounded operator from $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ to $\mathcal{B}^{1/2}(\Sigma)$.

Proposition 4.8. *Let $z \in \rho(H)$, $R(z) = (H - z)^{-1}$, $\tau_{(\cdot)}$ be the shift operator in* (4.13) *and $\varepsilon \in (0, \varepsilon_{ABC})$ with ε_{ABC} satisfying* (4.19)*. Then, for any $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ the relation*

$$C_\varepsilon(z)u(t) = \boldsymbol{t}_\Sigma \tau_{\varepsilon t} R(z)u \qquad \text{for a.e. } t \in (-1, 1) \tag{4.22}$$

holds in $L^2(\Sigma; \mathbb{C}^N)$ and $\operatorname{ran} C_\varepsilon(z) \subset \mathcal{B}^{1/2}(\Sigma)$. Moreover, the operators $C_\varepsilon(z)$ are uniformly bounded from $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ to $\mathcal{B}^{1/2}(\Sigma)$ and for any $r \in \left(0, \frac{1}{2}\right)$ one has

$$\|C_\varepsilon(z) - C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0} \leq C\varepsilon^{1/2-r}. \tag{4.23}$$

Proof. First, we show (4.22) for $u \in C_0^\infty(\mathbb{R}^\theta; \mathbb{C}^N)$. By density and continuity, this implies (4.22) for all $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$. Recall that $R(z) : H^s(\mathbb{R}^\theta; \mathbb{C}^N) \to H^{s+1}(\mathbb{R}^\theta; \mathbb{C}^N)$ is bounded for $s \in \mathbb{R}$; see Proposition 3.3 (iii). Hence, by the Sobolev embedding theorem $R(z)u$ is continuous for $u \in C_0^\infty(\mathbb{R}^\theta; \mathbb{C}^N)$ and the same is true for $\tau_{\varepsilon t}R(z)u$. Furthermore, as $\tau_{\varepsilon t}R(z)u \in H^1(\mathbb{R}^\theta; \mathbb{C}^N)$ we conclude with Proposition 2.3 (ii) and Proposition 4.2 for $t \in (-1, 1)$ and $x_\Sigma \in \Sigma$ that

$$\boldsymbol{t}_\Sigma \tau_{\varepsilon t} R(z)u(x_\Sigma) = \tau_{\varepsilon t} R(z)u(x_\Sigma) = \int_{\mathbb{R}^\theta} G_z(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y)u(y)dy = C_\varepsilon(z)u(t)(x_\Sigma).$$

Hence, (4.22) is true.

Next, we show the inclusion $\operatorname{ran} C_\varepsilon(z) \subset \mathcal{B}^{1/2}(\Sigma)$. Assume that $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$. Then, by the boundedness of the trace operator $\boldsymbol{t}_\Sigma : H^1(\mathbb{R}^\theta; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$, see Proposition 2.3 (ii), and Proposition 4.6 it follows that the function $\boldsymbol{t}_\Sigma \tau_{(\cdot)\varepsilon} R(z)u$ is continuous as a mapping from $(-1, 1)$ to $H^{1/2}(\Sigma; \mathbb{C}^N)$. In particular, $\boldsymbol{t}_\Sigma \tau_{(\cdot)\varepsilon} R(z)u$ is measurable as a mapping from $(-1, 1)$ to $H^{1/2}(\Sigma; \mathbb{C}^N)$. Using again the boundedness of the trace operator $\boldsymbol{t}_\Sigma : H^1(\mathbb{R}^\theta; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$ and the uniform boundedness of the shift operator in $H^1(\mathbb{R}^\theta; \mathbb{C}^N)$, see Proposition 2.3 (ii) and Proposition 4.4, respectively, we conclude

$$\int_{-1}^{1} \|\boldsymbol{t}_\Sigma \tau_{\varepsilon t} R(z)u\|^2_{H^{1/2}(\Sigma;\mathbb{C}^N)}\, dt \leq \int_{-1}^{1} C\|R(z)u\|^2_{H^1(\mathbb{R}^\theta;\mathbb{C}^N)}\, dt \leq C\|u\|^2_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)}$$

and therefore $\boldsymbol{t}_\Sigma \tau_{(\cdot)\varepsilon} R(z)u \in \mathcal{B}^{1/2}(\Sigma)$. Moreover, this also shows that $C_\varepsilon(z)$ is uniformly bounded from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $\mathcal{B}^{1/2}(\Sigma)$.

Finally, with Proposition 2.3 (ii), Proposition 4.4 and $C_0(z) = \mathfrak{J}\Phi_{\overline{z}}^* = \mathfrak{J}\boldsymbol{t}_\Sigma R(z)$, see Proposition 3.6 (iii), we have for $r \in \left(0, \frac{1}{2}\right)$ and $u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$

$$\begin{aligned}
\|C_\varepsilon(z)u - C_0(z)u\|_0^2 &= \|\boldsymbol{t}_\Sigma \tau_{\varepsilon(\cdot)} R(z)u - \mathfrak{J}\boldsymbol{t}_\Sigma R(z)u\|_0^2 \\
&= \int_{-1}^{1} \|\boldsymbol{t}_\Sigma(\tau_{\varepsilon t} - I)R(z)u\|_{L^2(\Sigma;\mathbb{C}^N)}^2 \, dt \\
&\le C\int_{-1}^{1} \|(\tau_{\varepsilon t} - I)R(z)u\|_{H^{r+1/2}(\mathbb{R}^\theta;\mathbb{C}^N)}^2 \, dt \\
&\le C\int_{-1}^{1} |\varepsilon t|^{1-2r} \|R(z)u\|_{H^1(\mathbb{R}^\theta;\mathbb{C}^N)}^2 \, dt \\
&\le C\int_{-1}^{1} \varepsilon^{1-2r} \|u\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)}^2 \, dt \\
&\le C\varepsilon^{1-2r} \|u\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)}^2,
\end{aligned}$$

which leads to (4.23). Therefore, all claims are shown. □

Using the convergence of $C_\varepsilon(z)$, it is not difficult to show the convergence of $A_\varepsilon(z)$. We define the natural candidate for the limit operator by

$$A_0(z) := \Phi_z \mathfrak{J}^* : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N). \tag{4.24}$$

Proposition 4.9. *Let $z \in \rho(H)$ and $\varepsilon \in (0, \varepsilon_{ABC})$ with ε_{ABC} satisfying (4.19). Then, for any $r \in (0, \frac{1}{2})$ one has*

$$\|A_\varepsilon(z) - A_0(z)\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\varepsilon^{1/2-r}$$

and, in particular, the operators $A_\varepsilon(z) : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ are uniformly bounded.

Proof. Let $\mathcal{I}_\varepsilon$, S_ε, and M_ε be the operators given by (4.5), (4.6), and (4.20), respectively. One verifies by a direct calculation using Proposition 2.12 (iii), (4.5), and (4.6) that $(\mathcal{I}_\varepsilon \mathcal{S}_\varepsilon)^* = M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1}$. Using this relation we conclude from (4.8) that

$$\begin{aligned}
\left(A_\varepsilon(z) M_\varepsilon^{-1}\right)^* &= M_\varepsilon^{-1}(A_\varepsilon(z))^* = M_\varepsilon^{-1}(R(z)U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon)^* \\
&= M_\varepsilon^{-1} M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(\overline{z}) = \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(\overline{z}) = C_\varepsilon(\overline{z}).
\end{aligned} \tag{4.25}$$

Moreover, $(A_0(z))^* = (\Phi_z \mathfrak{J}^*)^* = \mathfrak{J}\Phi_z^* = C_0(\overline{z})$. Hence, Lemma 4.7 and Proposition 4.8 yield

$$\begin{aligned}
&\big\|A_\varepsilon(z) - A_0(z)\big\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad = \|A_\varepsilon(z) M_\varepsilon^{-1}(M_\varepsilon - I) + A_\varepsilon(z) M_\varepsilon^{-1} - A_0(z)\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad \le C\varepsilon \|C_\varepsilon(\overline{z})\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0} + \|C_\varepsilon(\overline{z}) - C_0(\overline{z})\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0} \\
&\quad \le C\varepsilon^{1/2-r},
\end{aligned}$$

which is the claimed estimate. □

Next, we study the convergence of the operators $B_\varepsilon(z)$. We define the limit operator $B_0(z) : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma)$ which acts on $f \in \mathcal{B}^0(\Sigma)$ evaluated for a.e. $t \in (-1,1)$ as

$$B_0(z)f(t) := \frac{i}{2}(\alpha \cdot \nu) \int_{-1}^{1} \operatorname{sign}(t-s) f(s)\, ds + \mathcal{C}_z \int_{-1}^{1} f(s)\, ds, \tag{4.26}$$

where $\mathcal{C}_z : L^2(\Sigma;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N)$ is the operator defined in Definition 3.7. Using the mapping properties of $\mathcal{C}_z$ from Proposition 3.8 (i) it follows that $B_0(z)$ can also be regarded as a bounded operator in $\mathcal{B}^r(\Sigma)$ for any $r \in [-\frac{1}{2}, \frac{1}{2}]$. In the following proposition we show that $B_\varepsilon(z)$ converges to $B_0(z)$. The proof of this result is more complicated than the proofs of Proposition 4.8 and Proposition 4.9, and therefore some of the more technical calculations are shifted to Appendix B.

Proposition 4.10. *Let $z \in \rho(H)$ and $\varepsilon \in (0, \varepsilon_{ABC})$ with ε_{ABC} satisfying* (4.19). *Then, the operators $B_\varepsilon(z)$ are uniformly bounded in $\mathcal{B}^0(\Sigma)$ and for any $r \in (0, \frac{1}{2})$ one has*

$$\|B_\varepsilon(z) - B_0(z)\|_{1/2\to 0} \leq C\varepsilon^{1/2-r}.$$

Proof. The proof is split into several steps. Let Φ_z be as in Definition 3.5 and let $\tau^{\Omega_\pm}_{(\cdot)}$ be defined by (4.18). We introduce the auxiliary operators

$$\widetilde{B}_\varepsilon(z) := B_\varepsilon(z) M_\varepsilon^{-1} : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma), \tag{4.27}$$

which are, due to the properties of $B_\varepsilon(z)$ and M_ε in (4.8) and (4.20), bounded and act on $f \in \mathcal{B}^0(\Sigma)$ for a.e. $t \in (-1,1)$ and σ-a.e. $x_\Sigma \in \Sigma$ as

$$\widetilde{B}_\varepsilon(z) f(t)(x_\Sigma) = \int_{-1}^{1} \int_\Sigma G_z(x_\Sigma + \varepsilon t \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)) f(s)(y_\Sigma)\, d\sigma(y_\Sigma)\, ds. \tag{4.28}$$

Moreover, we define

$$\overline{B}_\varepsilon(z) : \mathcal{B}^{1/2}(\Sigma) \to \mathcal{B}^{1/2}(\Sigma)$$

acting on $f \in \mathcal{B}^{1/2}(\Sigma)$ for a.e. $t \in (-1,1)$ as

$$\overline{B}_\varepsilon(z) f(t) = \int_{-1}^{t} \boldsymbol{t}_\Sigma^- \tau^{\Omega_-}_{\varepsilon(t-s)} (\Phi_z f(s))_- ds + \int_{t}^{1} \boldsymbol{t}_\Sigma^+ \tau^{\Omega_+}_{\varepsilon(t-s)} (\Phi_z f(s))_+ \, ds. \tag{4.29}$$

First, in *Step 1* we show that $\overline{B}_\varepsilon(z)$ is bounded and converges to $B_0(z)$. Then, in *Step 2* we verify an alternative representation of $\overline{B}_\varepsilon(z)$. In *Step 3* we use Appendix B to compare $\overline{B}_\varepsilon(z)$ and $\widetilde{B}_\varepsilon(z)$, and show that $\widetilde{B}_\varepsilon(z)$ is uniformly bounded in ε. In *Step 4* we combine the results from *Step 1* to *Step 3* to conclude the claims of this proposition.

Step 1. First, we note that, due to Definition 2.13 and Proposition 4.6, the function $(-1,1)^2 \ni (t,s) \mapsto \Theta(\mp(t-s))\tau^{\Omega_\pm}_{\varepsilon(t-s)} \in \mathcal{L}(H^1(\Omega_\pm;\mathbb{C}^N), H^1(\Omega_\pm;\mathbb{C}^N))$ is measurable, where Θ is the Heaviside function. Hence, it follows from the text below Definition 2.13 that the integrands in (4.29) are measurable with respect to $(t,s) \in (-1,1)^2$. Moreover, by the mapping properties of $\boldsymbol{t}_\Sigma^\pm$, Φ_z and $\tau^{\Omega_\pm}_{(\cdot)}$ in Proposition 2.3, Proposition 3.6 and Corollary 4.5 respectively, the integrands are bounded by $C\|f(s)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)}$ for $(t,s) \in (-1,1)^2$. In particular, we conclude that for $f \in \mathcal{B}^{1/2}(\Sigma)$

$$\int_{-1}^1\int_{-1}^1 \left\|\Theta(t-s)\boldsymbol{t}_\Sigma^-\tau^{\Omega_-}_{\varepsilon(t-s)}(\Phi_z f(s))_- + \Theta(s-t)\boldsymbol{t}_\Sigma^+\tau^{\Omega_+}_{\varepsilon(t-s)}(\Phi_z f(s))_+\right\|^2_{H^{1/2}(\Sigma;\mathbb{C}^N)} dt ds$$

is finite. Thus, Fubini's theorem for Bochner integrals, see Proposition 2.15, yields the integrability of the integrands in (4.29) with respect to $s \in (-1,1)$ for a.e. $t \in (-1,1)$ and the measurability of $t \mapsto \overline{B}_\varepsilon(z)f(t)$. Furthermore, the bound for the integrands also implies $\|\overline{B}_\varepsilon(z)f\|_{1/2} \leq C\|f\|_{1/2}$ for $f \in \mathcal{B}^{1/2}(\Sigma)$. Hence, $\overline{B}_\varepsilon(z)$ is well-defined and uniformly bounded in $\mathcal{B}^{1/2}(\Sigma)$. We claim that

$$\|\overline{B}_\varepsilon(z) - B_0(z)\|_{1/2\to 0} \leq C\varepsilon^{1/2-r}. \tag{4.30}$$

To see this we remark that with Proposition 3.8 (ii) we have the pointwise representation

$$B_0(z)f(t) = \int_{-1}^t \boldsymbol{t}_\Sigma^-\left(\Phi_z f(s)\right)_- \, ds + \int_t^1 \boldsymbol{t}_\Sigma^+\left(\Phi_z f(s)\right)_+ \, ds$$

for a.e. $t \in (-1,1)$ and $f \in \mathcal{B}^{1/2}(\Sigma)$. Thus, $r \in \left(0,\frac{1}{2}\right)$ and direct estimates show

$$\begin{aligned}\|\overline{B}_\varepsilon(z)f - B_0(z)f\|_0^2 &= \int_{-1}^1 \Bigg\| \int_{-1}^t \boldsymbol{t}_\Sigma^-\big(\tau^{\Omega_-}_{\varepsilon(t-s)} - I\big)(\Phi_z f(s))_- ds \\ &\qquad + \int_t^1 \boldsymbol{t}_\Sigma^+\big(\tau^{\Omega_+}_{\varepsilon(t-s)} - I\big)(\Phi_z f(s))_+ ds \Bigg\|^2_{L^2(\Sigma;\mathbb{C}^N)} dt \\ &\leq \int_{-1}^1 \Bigg(\int_{-1}^t \left\|\boldsymbol{t}_\Sigma^-\big(\tau^{\Omega_-}_{\varepsilon(t-s)} - I\big)(\Phi_z f(s))_-\right\|_{H^r(\Sigma;\mathbb{C}^N)} ds \\ &\qquad + \int_t^1 \left\|\boldsymbol{t}_\Sigma^+\big(\tau^{\Omega_+}_{\varepsilon(t-s)} - I\big)(\Phi_z f(s))_+\right\|_{H^r(\Sigma;\mathbb{C}^N)} ds \Bigg)^2 dt.\end{aligned} \tag{4.31}$$

Employing Proposition 2.3, Proposition 3.6 (i) and Corollary 4.5 gives us for all $s,t \in (-1,1)$

$$\left\|\boldsymbol{t}_\Sigma^\pm\big(\tau^{\Omega_\pm}_{\varepsilon(t-s)} - I\big)(\Phi_z f(s))_\pm\right\|_{H^r(\Sigma;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}\|f(s)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)}.$$

Plugging this into (4.31) yields

$$\begin{aligned}
&\left\|\overline{B}_\varepsilon(z)f - B_0(z)f\right\|_0^2\\
&\quad\leq C\varepsilon^{2(1/2-r)}\int_{-1}^{1}\left(\int_{-1}^{t}\|f(s)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)}\,ds + \int_{t}^{1}\|f(s)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)}\,ds\right)^2 dt\\
&\quad\leq C\varepsilon^{2(1/2-r)}\int_{-1}^{1}\|f(s)\|^2_{H^{1/2}(\Sigma;\mathbb{C}^N)}\,ds = C\varepsilon^{2(1/2-r)}\|f\|^2_{1/2},
\end{aligned}$$

which implies (4.30).

Step 2. We show that the operator $\overline{B}_\varepsilon(z)$ in (4.29) has the alternative representation

$$\overline{B}_\varepsilon(z)f(t)(x_\Sigma) = \int_{-1}^{1}\int_\Sigma G_z(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) - y_\Sigma)f(s)(y_\Sigma)\,d\sigma(y_\Sigma)\,ds \tag{4.32}$$

for $f \in \mathcal{B}^{1/2}(\Sigma)$, a.e. $t \in (-1,1)$ and σ-a.e. $x_\Sigma \in \Sigma$. Let $f \in \mathcal{B}^{1/2}(\Sigma)$ and $t, s \in (-1,1)$ be fixed such that $t > s$. Note that the choice of $\varepsilon_{\mathrm{tub}}$ in Proposition 2.12 implies $\varepsilon_{ABC} \leq \frac{\varepsilon_{\mathrm{tub}}}{2} < \frac{\varepsilon_\iota}{2}$; cf. also Proposition 2.9. Hence, by Corollary 2.10 we have $x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) \in \Omega_-$ for all $x_\Sigma \in \Sigma$. Moreover, we conclude from the representation of Φ_z given in Definition 3.5 and the form of the integral kernel G_z, see (3.3)–(3.4), that $\Phi_z f(s)$ is continuous away from Σ. Thus, we have for σ-a.e. $x_\Sigma \in \Sigma$

$$\begin{aligned}
\boldsymbol{t}_\Sigma^-\tau^{\Omega_-}_{\varepsilon(t-s)}(\Phi_z f(s))_-(x_\Sigma) &= (\Phi_z f(s))(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma))\\
&= \int_\Sigma G_z(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) - y_\Sigma)f(s)(y_\Sigma)\,d\sigma(y_\Sigma).
\end{aligned}$$

Analogously, for $t < s$ and σ-a.e. $x_\Sigma \in \Sigma$

$$\begin{aligned}
\boldsymbol{t}_\Sigma^+\tau^{\Omega_+}_{\varepsilon(t-s)}(\Phi_z f(s))_+(x_\Sigma) &= (\Phi_z f(s))(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma))\\
&= \int_\Sigma G_z(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) - y_\Sigma)f(s)(y_\Sigma)\,d\sigma(y_\Sigma).
\end{aligned}$$

Combining the previous two equations yields

$$\begin{aligned}
\int_{-1}^{t}\boldsymbol{t}_\Sigma^-\tau^{\Omega_-}_{\varepsilon(t-s)}(\Phi_z f(s))_-(x_\Sigma)\,ds &+ \int_{t}^{1}\boldsymbol{t}_\Sigma^+\tau^{\Omega_+}_{\varepsilon(t-s)}(\Phi_z f(s))_+(x_\Sigma)\,ds\\
&= \int_{-1}^{1}\int_\Sigma G_z(x_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) - y_\Sigma)f(s)(y_\Sigma)\,d\sigma(y_\Sigma)\,ds.
\end{aligned}\tag{4.33}$$

Moreover, as the integrands on the right-hand side in (4.29) are Bochner integrable (cf. *Step 1*), Proposition 2.18 (iii) shows that the pointwise evaluation of the Bochner integrals in the definition of $\overline{B}_\varepsilon(z)$ in (4.29) coincides with (4.33), i.e.

$$\begin{aligned}\overline{B}_\varepsilon(z)f(t)(x_\Sigma) &= \left(\int_{-1}^{t} \boldsymbol{t}_\Sigma^- \tau^{\Omega_-}_{\varepsilon(t-s)}\left(\Phi_z f(s)\right)_-\, ds + \int_t^1 \boldsymbol{t}_\Sigma^+ \tau^{\Omega_+}_{\varepsilon(t-s)}\left(\Phi_z f(s)\right)_+\, ds\right)(x_\Sigma)\\ &= \int_{-1}^{t} \boldsymbol{t}_\Sigma^- \tau^{\Omega_-}_{\varepsilon(t-s)}(\Phi_z f(s))_-(x_\Sigma)\, ds + \int_t^1 \boldsymbol{t}_\Sigma^+ \tau^{\Omega_+}_{\varepsilon(t-s)}(\Phi_z f(s))_+(x_\Sigma)\, ds.\end{aligned}$$

This is exactly the claimed formula in (4.32).

Step 3. By the results in Appendix B the map $\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)$ admits an extension to a bounded operator from $\mathcal{B}^0(\Sigma)$ to $\mathcal{B}^{1/2}(\Sigma)$ and

$$\left\|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\right\|_{0\to 1/2} \leq C\varepsilon^{1/2}(1 + |\log(\varepsilon)|)^{1/2}. \tag{4.34}$$

Moreover, we claim that $\widetilde{B}_\varepsilon(z)$ is uniformly bounded in $\mathcal{B}^0(\Sigma)$. To see this, observe first that

$$\begin{aligned}\|\widetilde{B}_\varepsilon(z)\|_{1/2\to 1/2} &\leq \|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{1/2\to 1/2} + \|\overline{B}_\varepsilon(z)\|_{1/2\to 1/2}\\ &\leq \|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{0\to 1/2} + \|\overline{B}_\varepsilon(z)\|_{1/2\to 1/2}.\end{aligned}$$

Therefore, the estimate (4.34) and the uniform boundedness of $\overline{B}_\varepsilon(z)$ in $\mathcal{B}^{1/2}(\Sigma)$ shown in *Step 1* imply that $\widetilde{B}_\varepsilon(z)$ is also uniformly bounded in $\mathcal{B}^{1/2}(\Sigma)$. The same is true for $\widetilde{B}_\varepsilon(\overline{z})$ and hence also the antidual

$$(\widetilde{B}_\varepsilon(\overline{z}) \upharpoonright \mathcal{B}^{1/2}(\Sigma))' : \mathcal{B}^{-1/2}(\Sigma) \to \mathcal{B}^{-1/2}(\Sigma)$$

is uniformly bounded. Here, we used that $(\mathcal{B}^{1/2}(\Sigma))'$ can be identified with $\mathcal{B}^{-1/2}(\Sigma)$; see Proposition 2.21 (ii). We claim that $(\widetilde{B}_\varepsilon(\overline{z}) \upharpoonright \mathcal{B}^{1/2}(\Sigma))'$ is an extension of $\widetilde{B}_\varepsilon(z)$, that is,

$$\widetilde{B}_\varepsilon(z)f = (\widetilde{B}_\varepsilon(\overline{z}) \upharpoonright \mathcal{B}^{1/2}(\Sigma))'f, \qquad f \in \mathcal{B}^0(\Sigma). \tag{4.35}$$

In fact, the identities $\widetilde{B}_\varepsilon(\overline{z}) = B_\varepsilon(\overline{z})M_\varepsilon^{-1}$, (4.8), and $(\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon)^* = M_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}$, yield for the adjoint of $\widetilde{B}_\varepsilon(\overline{z})$ in $\mathcal{B}^0(\Sigma)$

$$\begin{aligned}(\widetilde{B}_\varepsilon(\overline{z}))^* &= \left(\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon R(\overline{z})U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon M_\varepsilon^{-1}\right)^*\\ &= \mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon R(z)U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon M_\varepsilon^{-1} = \widetilde{B}_\varepsilon(z)\end{aligned} \tag{4.36}$$

and hence Proposition 2.21 (ii) implies for $f \in \mathcal{B}^0(\Sigma)$ and $g \in \mathcal{B}^{1/2}(\Sigma)$

$$\begin{aligned}\left\langle(\widetilde{B}_\varepsilon(\overline{z}) \upharpoonright \mathcal{B}^{1/2}(\Sigma))'f, g\right\rangle_{\mathcal{B}^{-1/2}(\Sigma)\times\mathcal{B}^{1/2}(\Sigma)} &= \langle f, \widetilde{B}_\varepsilon(\overline{z})g\rangle_{\mathcal{B}^{-1/2}(\Sigma)\times\mathcal{B}^{1/2}(\Sigma)}\\ &= (f, \widetilde{B}_\varepsilon(\overline{z})g)_{\mathcal{B}^0(\Sigma)}\\ &= (\widetilde{B}_\varepsilon(z)f, g)_{\mathcal{B}^0(\Sigma)}\\ &= \langle\widetilde{B}_\varepsilon(z)f, g\rangle_{\mathcal{B}^{-1/2}(\Sigma)\times\mathcal{B}^{1/2}(\Sigma)},\end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{B}^{-1/2}(\Sigma)\times\mathcal{B}^{1/2}(\Sigma)}$ denotes the sesquilinear duality product; cf. Section 2.1 (iii). This implies (4.35) and since $(\widetilde{B}_\varepsilon(\overline{z}))'$ and $\widetilde{B}_\varepsilon(z)$ are both uniformly bounded in $\mathcal{B}^{-1/2}(\Sigma)$ and $\mathcal{B}^{1/2}(\Sigma)$, respectively, an interpolation argument, see Proposition 2.21 (i) and Section 2.1 (xiii), leads to the uniform boundedness of $\widetilde{B}_\varepsilon(z)$ in $\mathcal{B}^0(\Sigma)$.

Step 4. Using the results from *Step 1* to *Step 3* we now complete the proof of Proposition 4.10. Since $B_\varepsilon(z) = \widetilde{B}_\varepsilon(z)M_\varepsilon$, Lemma 4.7 and the uniform boundedness of $\widetilde{B}_\varepsilon(z)$ shown in *Step 3* imply the uniform boundedness of $B_\varepsilon(z)$ in $\mathcal{B}^0(\Sigma)$, proving the first claim of this proposition. Moreover, by applying Lemma 4.7, (4.30), (4.34) and the uniform boundedness of $\widetilde{B}_\varepsilon(z)$ in the space $\mathcal{B}^0(\Sigma)$ we obtain

$$\begin{aligned}
\|B_\varepsilon(z) - B_0(z)\|_{1/2\to 0} &= \|\widetilde{B}_\varepsilon(z)M_\varepsilon - B_0(z)\|_{1/2\to 0} \\
&\leq \|\widetilde{B}_\varepsilon(z)(M_\varepsilon - I)\|_{1/2\to 0} + \|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{1/2\to 0} + \|\overline{B}_\varepsilon(z) - B_0(z)\|_{1/2\to 0} \\
&\leq \|\widetilde{B}_\varepsilon(z)(M_\varepsilon - I)\|_{0\to 0} + \|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{0\to 1/2} + \|\overline{B}_\varepsilon(z) - B_0(z)\|_{1/2\to 0} \\
&\leq C\big(\varepsilon + \varepsilon^{1/2}(1 + |\log(\varepsilon)|)^{1/2} + \varepsilon^{1/2-r}\big) \\
&\leq C\varepsilon^{1/2-r}.
\end{aligned}$$

This is the claimed norm estimate and finishes the proof of this proposition. □

4.4 Properties of the limit operators $A_0(z)$, $B_0(z)$ and $C_0(z)$

After discussing the convergence properties of $A_\varepsilon(z)$, $B_\varepsilon(z)$ and $C_\varepsilon(z)$, we discuss in this section the limit operators $A_0(z)$, $B_0(z)$ and $C_0(z)$. In particular, we give in Proposition 4.14 conditions under which

$$(H - z)^{-1} - A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z)$$

is the resolvent of $H_{\widetilde{V}\delta_\Sigma}$, where $\widetilde{V} = \widetilde{V}(V)$ is a rescaled interaction matrix. This section contains parts of [14, Section 4] and [15].

To study the expression $(H - z)^{-1} - A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z)$ it is essential to investigate the operator $I + B_0(z)Vq$ and its inverse. This requires some technical preparations and we first introduce the operator

$$\begin{aligned}
T &: \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma), \\
Tf(t) &:= \frac{i}{2}\int_{-1}^{1} \operatorname{sign}(t - s)f(s)\,ds,
\end{aligned}$$

and the function

$$Q(t) := -\frac{1}{2} + \int_{-1}^{t} q(s)ds, \quad t \in [-1, 1], \tag{4.37}$$

where $q \in L^\infty((-1,1);\mathbb{R})$ is the function introduced in (4.1). The function Q satisfies $Q' = q$, $Q(-1) = -\frac{1}{2}$, and since $\int_{-1}^1 q(s)\,ds = 1$, also $Q(1) = -Q(-1) = \frac{1}{2}$. Moreover, for $r \in [0, \frac{1}{2}]$ the map T gives rise to a bounded operator in $\mathcal{B}^r(\Sigma)$ and

$$B_0(z) = T(\alpha \cdot \nu) + \mathfrak{J}\mathcal{C}_z\mathfrak{J}^* \tag{4.38}$$

with $\mathfrak{J}$ from (2.10).

Lemma 4.11. *Let q and V be as in* (4.1) *and* (4.2)*, let $r \in [0, \frac{1}{2}]$, and assume $\cos\bigl(\frac{(\alpha\cdot\nu)V}{2}\bigr)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. Then, the following is true:*

(i) *$I + T(\alpha\cdot\nu)Vq$ is boundedly invertible in $\mathcal{B}^r(\Sigma)$ and its inverse is given by* (4.40)*.*

(ii) *If $f \in \operatorname{ran}\mathfrak{J}$, i.e. f is independent of $t \in (-1,1)$, then the equation*

$$(I + T(\alpha\cdot\nu)Vq)^{-1}f(t) = \cos\bigl(\tfrac{(\alpha\cdot\nu)V}{2}\bigr)^{-1}\exp(-i(\alpha\cdot\nu)VQ(t))f(t)$$

holds for a.e. $t \in (-1,1)$.

Proof. (i) We show that the operator O defined in (4.40) below is the inverse of $I + T(\alpha\cdot\nu)Vq$. We start by fixing $r \in [0, \frac{1}{2}]$ and defining the operators

$$\begin{aligned} &\Xi : \mathcal{B}^r(\Sigma) \to H^r(\Sigma;\mathbb{C}^N),\\ &\Xi f := \frac{1}{2}\cos\bigl(\tfrac{(\alpha\cdot\nu)V}{2}\bigr)^{-1} i(\alpha\cdot\nu)V\int_{-1}^1 \exp\bigl(i(\alpha\cdot\nu)V(Q(s)-\tfrac{1}{2})\bigr)q(s)f(s)\,ds, \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} O &: \mathcal{B}^r(\Sigma) \to \mathcal{B}^r(\Sigma),\\ Of(t) &:= f(t) + \exp(-i(\alpha\cdot\nu)VQ(t))\Xi f\\ &\qquad - i(\alpha\cdot\nu)V\int_{-1}^t \exp\bigl(i(\alpha\cdot\nu)V(Q(s)-Q(t))\bigr)q(s)f(s)\,ds. \end{aligned} \tag{4.40}$$

We argue that Ξ and O are bounded and that $O = (I + T(\alpha\cdot\nu)Vq)^{-1}$. First, we verify that Ξ is well-defined and bounded. Let $f \in \mathcal{B}^r(\Sigma)$. Then, the integrand in (4.39) is measurable as a function from $(-1,1)$ to $H^r(\Sigma;\mathbb{C}^N)$ since the function $\bigl(\exp\bigl(i(\alpha\cdot\nu)V\bigl(Q(\cdot)-\frac{1}{2}\bigr)\bigr)q(\cdot)f(\cdot),\psi\bigr)_{H^r(\Sigma;\mathbb{C}^N)}$ is measurable for all $\psi \in H^r(\Sigma;\mathbb{C}^N)$; see Definition 2.13. In fact, the latter function is the pointwise limit of the sequence of measurable functions

$$\begin{aligned} t \mapsto &\sum_{k=0}^n \frac{\bigl(\bigl(i(\alpha\cdot\nu)V\bigl(Q(t)-\frac{1}{2}\bigr)\bigr)^k q(t)f(t),\psi\bigr)_{H^r(\Sigma;\mathbb{C}^N)}}{k!}\\ &\qquad = \sum_{k=0}^n \frac{\bigl(Q(t)-\frac{1}{2}\bigr)^k q(t)\bigl((i(\alpha\cdot\nu)V)^k f(t),\psi\bigr)_{H^r(\Sigma;\mathbb{C}^N)}}{k!}. \end{aligned}$$

Moreover, as $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1}, \alpha\cdot\nu, V \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ it follows from Proposition 2.2 (iii) that

$$\begin{aligned}
&\|\Xi f\|_{H^r(\Sigma;\mathbb{C}^N)} \\
&\quad = \frac{1}{2}\bigg\|\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} i(\alpha\cdot\nu)V\int_{-1}^{1}\exp\big(i(\alpha\cdot\nu)V\big(Q(s)-\tfrac{1}{2}\big)\big)q(s)f(s)\,ds\bigg\|_{H^r(\Sigma;\mathbb{C}^N)} \\
&\quad \leq C\int_{-1}^{1}\big\|\exp\big(i(\alpha\cdot\nu)V\big(Q(s)-\tfrac{1}{2}\big)\big)q(s)f(s)\big\|_{H^r(\Sigma;\mathbb{C}^N)}\,ds
\end{aligned}$$

and $\alpha\cdot\nu, V \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ also implies $\exp(i(\alpha\cdot\nu)V(Q(s)-\frac{1}{2})) \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ via the power series of the exponential function. Using $q \in L^\infty((-1,1);\mathbb{R})$ we conclude

$$\begin{aligned}
&\|\Xi f\|_{H^r(\Sigma;\mathbb{C}^N)} \\
&\quad \leq C\int_{-1}^{1}\big\|\exp\big((\alpha\cdot\nu)V(Q(s)-\tfrac{1}{2})\big)\big\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}\|q\|_{L^\infty((-1,1))}\|f(s)\|_{H^r(\Sigma;\mathbb{C}^N)}\,ds \\
&\quad \leq C\int_{-1}^{1}\|f(s)\|_{H^r(\Sigma;\mathbb{C}^N)}\,ds \\
&\quad \leq C\|f\|_r.
\end{aligned}$$

This shows that Ξ is well-defined and bounded. Analogously, one can check that O is well-defined and bounded. Hence, in order to show (i) it suffices to prove

$$(I+T(\alpha\cdot\nu)Vq)Of = O(I+T(\alpha\cdot\nu)Vq)f = f \tag{4.41}$$

for all $f \in \mathcal{B}^0(\Sigma)$. By Proposition 2.18 (iii) this is true, if for σ-a.e. $x_\Sigma \in \Sigma$ the relation

$$(I+T(\alpha\cdot\nu)Vq)Of(\cdot)(x_\Sigma) = O(I+T(\alpha\cdot\nu)Vq)f(\cdot)(x_\Sigma) = f(\cdot)(x_\Sigma)$$

holds a.e. on $(-1,1)$. For $f \in \mathcal{B}^0(\Sigma)$ we get from Proposition 2.18 (iii) that the function $(t,x_\Sigma)\mapsto f(t)(x_\Sigma)$ is in $L^2((-1,1)\times\Sigma;\mathbb{C}^N)$ and thus $f(\cdot)(x_\Sigma) \in L^2((-1,1);\mathbb{C}^N)$ for σ-a.e. x_Σ in Σ. We fix such an $x_\Sigma \in \Sigma$ and define $\varphi := f(\cdot)(x_\Sigma) \in L^2((-1,1);\mathbb{C}^N)$ and $A := (\alpha\cdot\nu(x_\Sigma))V(x_\Sigma)$. Then, we have for a.e. $t \in (-1,1)$

$$\begin{aligned}
&(I+T(\alpha\cdot\nu)Vq)Of(t)(x_\Sigma) \\
&= \varphi(t) + \exp(-iAQ(t))\Xi f(x_\Sigma) - iA\int_{-1}^{t}\exp\big(iA(Q(s)-Q(t))\big)q(s)\varphi(s)\,ds \\
&\quad + \frac{i}{2}\int_{-1}^{1}\operatorname{sign}(t-s)Aq(s)\bigg(\varphi(s)+\exp(-iAQ(s))\Xi f(x_\Sigma) \\
&\qquad\qquad - iA\int_{-1}^{s}\exp\big(iA(Q(r)-Q(s))\big)q(r)\varphi(r)\,dr\bigg)\,ds.
\end{aligned} \tag{4.42}$$

With a direct calculation we find that

$$
\begin{aligned}
&\frac{i}{2}\int_{-1}^{1} \operatorname{sign}(t-s) A \exp(-iAQ(s)) q(s) \Xi f(x_\Sigma)\, ds \\
&\quad = \frac{1}{2}\Big(\int_{-1}^{t} iA \exp(-iAQ(s)) q(s)\, ds - \int_{t}^{1} iA \exp(-iAQ(s)) q(s)\, ds \Big) \Xi f(x_\Sigma) \\
&\quad = -\exp(-iAQ(t)) \Xi f(x_\Sigma) + \frac{1}{2}\big(\exp\big(-i\tfrac{A}{2}\big) + \exp\big(i\tfrac{A}{2}\big)\big)\Xi f(x_\Sigma) \\
&\quad = -\exp(-iAQ(t)) \Xi f(x_\Sigma) + \cos\big(\tfrac{A}{2}\big)\Xi f(x_\Sigma).
\end{aligned}
$$

Furthermore, integration by parts gives us

$$
\begin{aligned}
&-\frac{i}{2}\int_{-1}^{1} \operatorname{sign}(t-s) A q(s) iA \int_{-1}^{s} \exp\big(iA(Q(r)-Q(s))\big) q(r)\varphi(r)\, dr\, ds \\
&\quad = \frac{i}{2} A \int_{-1}^{t} \frac{d}{ds}\big(\exp(-iAQ(s))\big) \int_{-1}^{s} \exp(iAQ(r)) q(r)\varphi(r)\, dr\, ds \\
&\qquad - \frac{i}{2} A \int_{t}^{1} \frac{d}{ds}\big(\exp(-iAQ(s))\big) \int_{-1}^{s} \exp(iAQ(r)) q(r)\varphi(r)\, dr\, ds \\
&\quad = iA \exp(-iAQ(t)) \int_{-1}^{t} \exp(iAQ(r)) q(r)\varphi(r)\, dr \\
&\qquad - \frac{i}{2} A \int_{-1}^{1} \exp(iA(Q(r) - \tfrac{1}{2})) q(r)\varphi(r)\, dr - \frac{i}{2} A \int_{-1}^{1} \operatorname{sign}(t-s) q(s)\varphi(s)\, ds \\
&\quad = iA \exp(-iAQ(t)) \int_{-1}^{t} \exp(iAQ(r)) q(r)\varphi(r)\, dr \\
&\qquad - \frac{i}{2} A \int_{-1}^{1} \operatorname{sign}(t-s) q(s)\varphi(s)\, ds - \cos\big(\tfrac{A}{2}\big)\Xi f(x_\Sigma).
\end{aligned}
$$

A combination of the last two calculations with (4.42) yields

$$(I + T(\alpha\cdot\nu)Vq)Of(t)(x_\Sigma) = \varphi(t).$$

One verifies in a very similar way that O is also the left inverse of $I + T(\alpha\cdot\nu)Vq$. Consequently, (4.41) is true.

(ii) Let $f \in \operatorname{ran}\mathfrak{J}$, that is, f is independent of $t \in (-1,1)$. Instead of inserting f in (4.40) we find it more convenient and easier to verify this claim directly by showing

$$(I + T(\alpha\cdot\nu)Vq)\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)f = f.$$

Similarly as above it suffices to prove

$$(I + T(\alpha\cdot\nu)Vq)\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)f(\cdot)(x_\Sigma) = f(\cdot)(x_\Sigma)$$

for σ-a.e. $x_\Sigma \in \Sigma$ a.e. on $(-1,1)$. Thus, we again fix $x_\Sigma \in \Sigma$ and use the same abbreviations as in the proof of (i). Since here f is constant with respect to t, $\varphi = f(t)(x_\Sigma)$ is also independent of t. We then compute

$$\begin{aligned}
&(I + T(\alpha\cdot\nu)Vq)\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)f(t)(x_\Sigma)\\
&\quad= \cos\big(\tfrac{A}{2}\big)^{-1}\exp(-iAQ(t))\varphi + \frac{i}{2}\int_{-1}^{1}\operatorname{sign}(t-s)Aq(s)\cos\big(\tfrac{A}{2}\big)^{-1}\exp(-iAQ(s))\varphi\, ds\\
&\quad= \cos\big(\tfrac{A}{2}\big)^{-1}\exp(-iAQ(t))\varphi - \frac{\cos\big(\tfrac{A}{2}\big)^{-1}}{2}\int_{-1}^{1}\operatorname{sign}(t-s)\frac{d}{ds}\exp(-iAQ(s))\, ds\varphi\\
&\quad= \cos\big(\tfrac{A}{2}\big)^{-1}\exp(-iAQ(t))\varphi\\
&\qquad\qquad - \frac{\cos\big(\tfrac{A}{2}\big)^{-1}}{2}\Big(2\exp(-iAQ(t)) - \exp(i\tfrac{A}{2}) - \exp(-i\tfrac{A}{2})\Big)\varphi\\
&\quad= \varphi = f(t)(x_\Sigma)
\end{aligned}$$

for a.e. $t \in (-1,1)$, which shows (ii). □

In the next lemma we study relations connecting the interaction matrix V from (4.2) and the matrix $\widetilde{V} = VS$ with

$$S := \operatorname{sinc}\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \tag{4.43}$$

if $\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. We call S the *scaling matrix.*

Lemma 4.12. *Let $z \in \rho(H)$, q and V be as in* (4.1) *and* (4.2)*, respectively, assume $\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and set $\widetilde{V} = VS$, where S is the scaling matrix from* (4.43)*. Then, the following is true:*

(i) *$S, \widetilde{V} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and, in particular, the multiplication by $\widetilde{V}$ gives rise to a bounded operator in $H^r(\Sigma;\mathbb{C}^N)$ for $r \in [0,1]$.*

(ii) *$\mathfrak{J}^* q\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)\mathfrak{J} = S$.*

(iii) *$(I + B_0(z)Vq)(I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{J} = \mathfrak{J}(I + \mathcal{C}_z\widetilde{V})$.*

Proof. (i) From $\alpha\cdot\nu, V \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ we conclude, using the power series of sinc and Proposition 2.2 (iv), $\operatorname{sinc}\big(\tfrac{(\alpha\cdot\nu)V}{2}\big) \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. Furthermore, the assumption $\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and Proposition 2.2 (iv) imply $S \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and $\widetilde{V} = VS \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. Since the multiplication by any $B \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ gives rise to a bounded operator in $H^r(\Sigma;\mathbb{C}^N)$ for $r \in [0,1]$, the same is true for $\widetilde{V}$; cf. Proposition 2.2 (iii).

(ii) Recall that $\mathfrak{J}$ is defined by (2.10) and that its adjoint acts as

$$\mathfrak{J}^* f = \int_{-1}^{1} f(t)\,dt, \qquad f \in \mathcal{B}^0(\Sigma).$$

As in the proof of the previous lemma we use the abbreviation $A = (\alpha \cdot \nu)V$. Then, with Proposition 2.18 (iii) we get for $\psi \in L^2(\Sigma; \mathbb{C}^N)$

$$\begin{aligned}\mathfrak{J}^* q \cos\big(\tfrac{A}{2}\big)^{-1} \exp(-iAQ)\mathfrak{J}\psi &= \int_{-1}^{1} \cos\big(\tfrac{A}{2}\big)^{-1} \exp(-iAQ(s))q(s)(\mathfrak{J}\psi)(s)\,ds \\ &= \cos\big(\tfrac{A}{2}\big)^{-1} \int_{-1/2}^{1/2} \exp(-iAr)\,dr\psi \\ &= \cos\big(\tfrac{A}{2}\big)^{-1} \int_{0}^{1/2} 2\cos(Ar)\,dr\psi.\end{aligned}$$

Using the power series of sinc one verifies $\int_0^{1/2} 2\cos(Ar)\,dr = \operatorname{sinc}\big(\tfrac{A}{2}\big)$ and hence

$$\mathfrak{J}^* q \cos\big(\tfrac{A}{2}\big)^{-1} \exp(-iAQ)\mathfrak{J}\psi = \operatorname{sinc}\big(\tfrac{A}{2}\big)\cos\big(\tfrac{A}{2}\big)^{-1}\psi = S\psi$$

for $\psi \in L^2(\Sigma; \mathbb{C}^N)$. This shows (ii).

(iii) For $\psi \in L^2(\Sigma; \mathbb{C}^N)$ item (ii) from the current lemma, (4.38) and Lemma 4.11 (ii) imply

$$\begin{aligned}&(I + B_0(z)Vq)(I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{J}\psi \\ &\quad= \mathfrak{J}\psi + \mathfrak{J}\mathcal{C}_z\mathfrak{J}^* Vq(I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{J}\psi \\ &\quad= \mathfrak{J}\psi + \mathfrak{J}\mathcal{C}_z\mathfrak{J}^* Vq\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)\mathfrak{J}\psi \\ &\quad= \mathfrak{J}\psi + \mathfrak{J}\mathcal{C}_z VS\psi \\ &\quad= \mathfrak{J}(I + \mathcal{C}_z\widetilde{V})\psi.\end{aligned}$$

□

Proposition 4.13. *Let $z \in \rho(H)$, $r \in [0, \frac{1}{2}]$, q and V be as in (4.1) and (4.2), assume $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N})$ and set $\widetilde{V} = VS$. Moreover, assume that $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^r(\Sigma; \mathbb{C}^N)$. Then, the operator $I + B_0(z)Vq$ is continuously invertible in the space $\mathcal{B}^r(\Sigma)$ and*

$$(I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{J}(I + \mathcal{C}_z\widetilde{V})^{-1} = (I + B_0(z)Vq)^{-1}\mathfrak{J}. \tag{4.44}$$

Proof. The operator $B_0(z)$ is according to the text below (4.26) bounded in $\mathcal{B}^r(\Sigma)$. Moreover, as $V \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N})$ and $q \in L^\infty((-1,1))$, V as well as q induce by Proposition 2.2 (iii) and (2.9) also bounded operators in $\mathcal{B}^r(\Sigma)$. Hence, $I + B_0(z)Vq$

is bounded in $\mathcal{B}^r(\Sigma)$ and therefore it suffices to show that $I + B_0(z)Vq$ is bijective in $\mathcal{B}^r(\Sigma)$.

Let us start with the injectivity. To do so, we use the representation of $B_0(z)$ given by (4.38) and assume

$$(I + B_0(z)Vq)f = (I + T(\alpha \cdot \nu)Vq)f + \mathfrak{I}\mathcal{C}_z V \mathfrak{I}^* q f = 0$$

for a $f \in \mathcal{B}^r(\Sigma)$. Applying the operator $(I + T(\alpha \cdot \nu)Vq)^{-1}$ yields

$$f + (I + T(\alpha \cdot \nu)Vq)^{-1}\mathfrak{I}\mathcal{C}_z V \mathfrak{I}^* q f = 0.$$

Using Lemma 4.11 (ii) gives us

$$f + \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \exp(-i(\alpha \cdot \nu)VQ)\mathfrak{I}\mathcal{C}_z V \mathfrak{I}^* q f = 0. \tag{4.45}$$

By applying $\mathfrak{I}^* q$ and Lemma 4.12 (ii) we obtain

$$\begin{aligned}\mathfrak{I}^* q f + \mathfrak{I}^* q \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} &\exp(-i(\alpha \cdot \nu)VQ)\mathfrak{I}\mathcal{C}_z V \mathfrak{I}^* q f \\ &= (I + S\mathcal{C}_z V)\mathfrak{I}^* q f = 0.\end{aligned} \tag{4.46}$$

Since $I + \mathcal{C}_z \widetilde{V}$ is continuously invertible in $H^r(\Sigma; \mathbb{C}^N)$ and $\widetilde{V} = VS$, $I + S\mathcal{C}_z V$ is by Proposition 2.29 also continuously invertible in $H^r(\Sigma; \mathbb{C}^N)$. Hence, applying $(I + S\mathcal{C}_z V)^{-1}$ to (4.46) yields $\mathfrak{I}^* q f = 0$ and thus (4.45) shows $f = 0$.

Next, we show the surjectivity. Let $g \in \mathcal{B}^r(\Sigma)$. We set $f_g = (I+T(\alpha\cdot\nu)Vq)^{-1}(g+\mathfrak{I}\psi)$, where

$$\psi = -(I + \mathcal{C}_z \widetilde{V})^{-1}\mathcal{C}_z \mathfrak{I}^* V q (I + T(\alpha \cdot \nu)Vq)^{-1} g.$$

Applying $(I + B_0(z)Vq) = I + (T(\alpha \cdot \nu) + \mathfrak{I}\mathcal{C}_z\mathfrak{I}^*)Vq$ to f_g and Lemma 4.12 (iii) yield

$$\begin{aligned}(I + B_0(z)Vq)f_g &= g + \mathfrak{I}\mathcal{C}_z\mathfrak{I}^* V q (I + T(\alpha \cdot \nu)Vq)^{-1} g \\ &\qquad + (I + B_0(z)Vq)(I + T(\alpha \cdot \nu)Vq)^{-1}\mathfrak{I}\psi \\ &= g + \mathfrak{I}\mathcal{C}_z\mathfrak{I}^* V q (I + T(\alpha \cdot \nu)Vq)^{-1} g \\ &\qquad + \mathfrak{I}(I + \mathcal{C}_z \widetilde{V})\psi \\ &= g,\end{aligned}$$

which shows that $I + B_0(z)Vq$ is continuously invertible in $\mathcal{B}^r(\Sigma)$. Applying the operators $(I + \mathcal{C}_z \widetilde{V})^{-1}$ and $(I + B_0(z)Vq)^{-1}$ from the left and from the right to Lemma 4.12 (iii), respectively, yields (4.44). □

Having provided all these preliminary results, we are well-equipped to prove the main result of this section, which is a resolvent formula for $H_{\widetilde{V}\delta_\Sigma}$ in terms of the operators $A_0(z)$, $B_0(z)$ and $C_0(z)$.

Proposition 4.14. *Let $z \in \rho(H)$, $R(z) = (H-z)^{-1}$ be the resolvent of the free Dirac operator, q and V be as in* (4.1) *and* (4.2)*, assume* $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ *and set* $\widetilde{V} = VS$*. Moreover, assume that* $I + \mathcal{C}_z\widetilde{V}$ *is continuously invertible in* $H^{1/2}(\Sigma;\mathbb{C}^N)$*. Then,* $I + B_0(z)Vq$ *is continuously invertible in* $\mathcal{B}^{1/2}(\Sigma)$*,* $H_{\widetilde{V}\delta_\Sigma}$ *is closed,* $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$ *and the formula*

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = R(z) - A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z)$$

holds.

Proof. Since $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$, Proposition 3.14 implies that $H_{\widetilde{V}\delta_\Sigma}$ is closed, $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$ and

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = R(z) - \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}. \tag{4.47}$$

Proposition 4.13 and the assumption $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ show that the operator $I + B_0(z)Vq$ is continuously invertible in $\mathcal{B}^{1/2}(\Sigma)$. Hence, by (4.21), (4.44) and Lemma 4.11 (ii) we get for $v \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$

$$\begin{aligned}
&(I + B_0(z)Vq)^{-1}C_0(z)v \\
&\qquad = (I + T(\alpha\cdot\nu)Vq)^{-1}\mathfrak{J}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}v \\
&\qquad = \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)\mathfrak{J}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}v.
\end{aligned} \tag{4.48}$$

With $A_0(z) = \Phi_z\mathfrak{J}^*$, Lemma 4.12 (ii), and $VS = \widetilde{V}$ we conclude

$$\begin{aligned}
&A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z)v \\
&\qquad = \Phi_zV\mathfrak{J}^*q\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)VQ)\mathfrak{J}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}v \\
&\qquad = \Phi_zVS(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}v \\
&\qquad = \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi^*_{\overline{z}}v.
\end{aligned}$$

Inserting this observation into (4.47) yields the claimed resolvent formula. □

4.5 Convergence conditions for Dirac operators with general strongly localized potentials

Now, we are in a position to state the first main result of this thesis, which provides sufficient conditions for the norm resolvent convergence of H_{V_ε}.

Theorem 4.15. *Let q and V be as in* (4.1) *and* (4.2)*,* $\varepsilon_{ABC} > 0$ *as in* (4.19)*,* V_ε *be defined by* (4.3) *and assume that for some* $z \in \rho(H)$ *the following conditions are fulfilled:*

(i) *There exists an* $\varepsilon_{\mathrm{conv}} \in (0, \varepsilon_{ABC}]$ *such that the inverse* $(I + B_\varepsilon(z)Vq)^{-1}$ *exists for* $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$ *and is uniformly bounded in* $\mathcal{B}^0(\Sigma)$.

(ii) $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma; \mathbb{C}^{N\times N})$.

(iii) *The operator* $I + \mathcal{C}_z\widetilde{V}$ *(*$\widetilde{V} = VS$ *with* S *from (4.43)) is bijective in* $H^{1/2}(\Sigma; \mathbb{C}^N)$.

Then, the operator $H_{\widetilde{V}\delta_\Sigma}$ *is self-adjoint,* $z \in \rho(H_{\widetilde{V}\delta_\Sigma}) \cap \rho(H_{V_\varepsilon})$ *for all* $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$ *and for any* $r \in (0, \frac{1}{2})$ *exists a* $C > 0$ *such that*

$$\big\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\varepsilon^{1/2-r} \tag{4.49}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$. *In particular,* H_{V_ε} *converges to* $H_{\widetilde{V}\delta_\Sigma}$ *in the norm resolvent sense as* $\varepsilon \to 0$.

Proof. Throughout this proof we assume $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$. By Proposition 4.14 we have that $I + B_0(z)Vq$ is continuously invertible in $B^{1/2}(\Sigma)$, $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$ and

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = (H - z)^{-1} - A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z).$$

The assumptions and Proposition 4.1 (ii) guarantee $z \in \rho(H_{V_\varepsilon})$ as well as

$$(H_{V_\varepsilon} - z)^{-1} = (H - z)^{-1} - A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}C_\varepsilon(z).$$

Subtracting the above two equations yields

$$\begin{aligned}
(H_{V_\varepsilon}-z)^{-1} &- (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\\
&= - A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}C_\varepsilon(z) + A_0(z)Vq(I + B_0(z)Vq)^{-1}C_0(z)\\
&= - A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}(C_\varepsilon(z) - C_0(z))\\
&\quad - A_\varepsilon(z)Vq\big((I + B_\varepsilon(z)Vq)^{-1} - (I + B_0(z)Vq)^{-1}\big)C_0(z)\\
&\quad - (A_\varepsilon(z) - A_0(z))Vq(I + B_0(z)Vq)^{-1}C_0(z).
\end{aligned} \tag{4.50}$$

In the following we use the uniform boundedness of $A_\varepsilon(z) : \mathcal{B}^0(\Sigma) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and $(I + B_\varepsilon(z)Vq)^{-1} : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma)$; cf. Proposition 4.9 and assumption (i). Employing this and Proposition 4.8 we see that

$$\begin{aligned}
\big\|A_\varepsilon(z)Vq(I + B_\varepsilon(z)Vq)^{-1}&(C_\varepsilon(z) - C_0(z))\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\le C\|C_\varepsilon(z) - C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0}\\
&\le C\varepsilon^{1/2-r}.
\end{aligned} \tag{4.51}$$

Since $C_0(z) : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to \mathcal{B}^{1/2}(\Sigma)$ and $(I + B_0(z)Vq)^{-1} : \mathcal{B}^{1/2}(\Sigma) \to \mathcal{B}^{1/2}(\Sigma)$ are bounded, see (4.21) and Proposition 4.14, we get from Proposition 4.10

$$\begin{aligned}
&\big\| A_\varepsilon(z)Vq \left((I + B_\varepsilon(z)Vq)^{-1} - (I + B_0(z)Vq)^{-1} \right) C_0(z) \big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad \le C \big\| (I + B_\varepsilon(z)Vq)^{-1} - (I + B_0(z)Vq)^{-1} \big\|_{1/2 \to 0} \\
&\quad \le C \big\| (I + B_\varepsilon(z)Vq)^{-1} \big\|_{0 \to 0} \\
&\qquad\qquad \cdot \| (B_\varepsilon(z) - B_0(z))Vq \|_{1/2 \to 0} \big\| (I + B_0(z)Vq)^{-1} \big\|_{1/2 \to 1/2} \\
&\quad \le C\varepsilon^{1/2-r}.
\end{aligned} \tag{4.52}$$

Eventually, in a similar way as in (4.51), we find with Proposition 4.9 that

$$\begin{aligned}
&\big\| (A_\varepsilon(z) - A_0(z))Vq(I + B_0(z)Vq)^{-1} C_0(z) \big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad \le C \| A_\varepsilon(z) - A_0(z) \|_{0 \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad \le C\varepsilon^{1/2-r}.
\end{aligned} \tag{4.53}$$

Combining (4.51)–(4.53) with (4.50) shows (4.49).

It remains to prove the self-adjointness of $H_{\widetilde{V}\delta_\Sigma}$. Let us first consider the case $z \in \mathbb{R}$. In this case $(H_{\widetilde{V}\delta_\Sigma} - z)^{-1}$ is a bounded self-adjoint operator with the dense range $\operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$; cf. Lemma 3.13. Thus,

$$H_{\widetilde{V}\delta_\Sigma} = ((H_{\widetilde{V}\delta_\Sigma} - z)^{-1})^{-1} + z$$

is also self-adjoint. Now, let $z \in \mathbb{C} \setminus \mathbb{R}$. Since $(H_{V_\varepsilon} - z)^{-1}$ converges in the the operator norm to $(H_{\widetilde{V}\delta_\Sigma} - z)^{-1}$, the adjoint resolvent $(H_{V_\varepsilon} - \overline{z})^{-1}$ converges also in the operator norm to $((H_{\widetilde{V}\delta_\Sigma} - z)^{-1})^*$. Furthermore, $\operatorname{ran}(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ is dense in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ by Lemma 3.13. Hence, it follows from [62, Theorem VIII.22] that there exists a self-adjoint operator $\widetilde{H}$ such that $z \in \rho(\widetilde{H})$ and $(\widetilde{H} - z)^{-1} = (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}$. This implies $\widetilde{H} = H_{\widetilde{V}\delta_\Sigma}$ and therefore $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and $z \in \rho(H_{\widetilde{V}\delta_\Sigma})$. □

To make the conditions of Theorem 4.15 more tangible, we show in the upcoming corollary that these conditions are satisfied if $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small.

Corollary 4.16. *Let q and V be as in (4.1) and (4.2), $\varepsilon_{ABC} > 0$ as in (4.19), V_ε be defined by (4.3) and $z \in \rho(H)$. If $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small, then the operator $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint, $z \in \rho(H_{\widetilde{V}\delta_\Sigma}) \cap \rho(H_{V_\varepsilon})$ for all $\varepsilon \in (0, \varepsilon_{ABC})$, and for any $r \in (0, \frac{1}{2})$ exists a $C > 0$ such that*

$$\big\| (H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1} \big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon_{ABC})$. In particular, H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Proof. Let us shortly check that the conditions of Theorem 4.15 are fulfilled. We start with (i). By Proposition 2.2 (iii), Proposition 4.10 and the comments below (2.9) we have

$$\sup_{\varepsilon\in(0,\varepsilon_{ABC})} \|B_\varepsilon(z)Vq\|_{0\to 0} \leq C\|q\|_{L^\infty((-1,1))}\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \sup_{\varepsilon\in(0,\varepsilon_{ABC})} \|B_\varepsilon(z)\|_{0\to 0} < 1$$

if $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small, i.e. (i) of Theorem 4.15 is fulfilled if we set $\varepsilon_{\mathrm{conv}} = \varepsilon_{ABC}$ and $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small. For condition (ii) we use the power series of cos and cosh to estimate

$$\Big\|\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big) - I_N\Big\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \leq \cosh\Big(\tfrac{\|(\alpha\cdot\nu)V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}}{2}\Big) - 1.$$

In particular, if $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is small enough, then $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)$ is invertible in $W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$ and

$$\Big\|\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1}\Big\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \leq \frac{1}{2-\cosh\Big(\tfrac{\|(\alpha\cdot\nu)V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}}{2}\Big)}.$$

Now, let us turn to (iii). Using $\widetilde{V} = VS$, (4.43), the power series of sinc and the estimate from above we obtain

$$\begin{aligned}
&\|\widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \\
&\quad\leq \|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} 2\frac{\sinh\Big(\tfrac{\|(\alpha\cdot\nu)V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}}{2}\Big)}{\|(\alpha\cdot\nu)V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}\Big(2-\cosh\Big(\tfrac{\|(\alpha\cdot\nu)V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}}{2}\Big)\Big)}.
\end{aligned}$$

Hence,

$$\|\mathcal{C}_z\widetilde{V}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \leq C\|\mathcal{C}_z\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)}\|\widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$$

is smaller than one if $\|V\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}$ is sufficiently small. In particular, in this case $I + \mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$, i.e. (iii) of Theorem 4.15 is also fulfilled. □

5 An explicit convergence condition for Dirac operators with strongly localized electrostatic and Lorentz scalar potentials

In Theorem 4.15 we provided conditions, which imply the norm resolvent convergence of H_{V_ε} as $\varepsilon \to 0$, with H_{V_ε} as in (4.4). Unfortunately, the conditions in Theorem 4.15 and Corollary 4.16 are not explicit and it may be hard to verify them. The aim of this chapter is to show that under the assumptions

$$q \in L^\infty((-1,1);[0,\infty)) \quad \text{such that} \quad \int_{-1}^{1} q(t)dt = 1 \tag{5.1}$$

and

$$V = \eta I_N + \tau\beta \quad \text{with} \quad \eta,\tau \in C_b^1(\Sigma;\mathbb{R}), \tag{5.2}$$

one can simplify the three conditions in Theorem 4.15 to the explicit and simple condition

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2. \tag{5.3}$$

We focus on interaction matrices of the form $V = \eta I_N + \tau\beta$, which model electrostatic and Lorentz scalar interactions, as they are the most prevalent interaction matrices in literature; cf. Section 3.3. In this situation we get with the help of (3.1) the identity $((\alpha\cdot\nu)V)^2 = dI_N$. Hence, the power series expansions of cos and sinc give us

$$\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big) = \cos\big(\tfrac{\sqrt{d}}{2}\big) I_N \quad \text{and} \quad \operatorname{sinc}\big(\tfrac{(\alpha\cdot\nu)V}{2}\big) = \operatorname{sinc}\big(\tfrac{\sqrt{d}}{2}\big) I_N. \tag{5.4}$$

This and (5.3) immediately imply $\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W_\infty^1(\Sigma;\mathbb{C}^{N\times N})$, i.e. (ii) in Theorem 4.15 is fulfilled. Moreover, in this case the scaling matrix, see (4.43), is given by

$$S = \operatorname{sinc}\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)\cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} = \frac{\operatorname{sinc}\big(\frac{\sqrt{d}}{2}\big)}{\cos\big(\frac{\sqrt{d}}{2}\big)} I_N = \operatorname{tanc}\big(\tfrac{\sqrt{d}}{2}\big) I_N, \tag{5.5}$$

where tanc is the function from item (xx) of Section 2.1, and therefore

$$\widetilde{V} = VS = \widetilde{\eta} I_N + \widetilde{\tau}\beta \quad \text{with} \quad (\widetilde{\eta},\widetilde{\tau}) = \operatorname{tanc}\big(\tfrac{\sqrt{d}}{2}\big)(\eta,\tau).$$

Thus,

$$\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4\tan\big(\tfrac{\sqrt{d}}{2}\big)^2 \tag{5.6}$$

and by (5.3)

$$\inf_{x_\Sigma\in\Sigma} |\widetilde{d}(x_\Sigma) - 4| > 0.$$

Now, Proposition 3.15 shows that $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and that $I+\mathcal{C}_z\widetilde{V}$ is continuously invertible in $H^{1/2}(\Sigma;\mathbb{C}^N)$ for $z\in\mathbb{C}\setminus\mathbb{R}$. In particular, (ii) of Theorem 4.15 is also fulfilled for $z\in\mathbb{C}\setminus\mathbb{R}$. Thus, in order to prove the norm resolvent convergence of H_{V_ε} one only has to guarantee that (i) of Theorem 4.15 is satisfied for $z\in\mathbb{C}\setminus\mathbb{R}$, i.e. it is suffices to show that the inverse of $I+B_\varepsilon(z)Vq$ is uniformly bounded in $\mathcal{B}^0(\Sigma)$.

We proceed in the current chapter as follows: First, we show in Section 5.1 the uniform boundedness of $(I+B_\varepsilon(z)Vq)^{-1}$ in $\mathcal{B}^0(\Sigma)$ if Σ is a rotated C_b^2-graph. Afterwards, we use in the proof of the main theorem, Theorem 5.20, a partition of unity to reduce the general case of a special C^2-surface to the case of a rotated C_b^2-graph. However, in this case the proof of the norm resolvent convergence follows immediately from the comments above and Section 5.1. Before we continue, let us mention that this chapter is based on [15].

5.1 Analysis of $I+B_\varepsilon(z)Vq$ for rotated C_b^2-graphs

In this section we show that if Σ is a rotated C_b^2-graph and if $\sup_{x_\Sigma\in\Sigma} d(x_\Sigma)<\frac{\pi^2}{4}$, $d=\eta^2-\tau^2$, then (i) of Theorem 4.15 is fulfilled, i.e. $(I+B_\varepsilon(z)Vq)^{-1}$ is uniformly bounded in $\mathcal{B}^0(\Sigma)$. To prove this result a very careful and deep analysis of $I+B_\varepsilon(z)Vq$ is necessary. We do this by meticulously studying $I+B_\varepsilon(z)Vq$ in the case where Σ is a hyperplane and η and τ are constant in Section 5.1.1. Then, we use a parameter dependent partition of unity in Section 5.1.2 to transfer the results to the case where Σ is a rotated C_b^2-graph and $\eta,\tau\in C_b^1(\Sigma;\mathbb{R})$.

Recall the bounded operators $\widetilde{B}_\varepsilon(z):\mathcal{B}^0(\Sigma)\to\mathcal{B}^0(\Sigma)$ and $\overline{B}_\varepsilon(z):\mathcal{B}^{1/2}(\Sigma)\to\mathcal{B}^{1/2}(\Sigma)$ from (4.27) and (4.29), respectively. By (B.1) the difference of $\widetilde{B}_\varepsilon(z)$ and $\overline{B}_\varepsilon(z)$ can be extended to a bounded operator mapping from $\mathcal{B}^0(\Sigma)$ to $\mathcal{B}^{1/2}(\Sigma)$. In particular, this extension acts also as a bounded operator in $\mathcal{B}^0(\Sigma)$. Thus,

$$\overline{B}_\varepsilon(z)=\widetilde{B}_\varepsilon(z)+(\overline{B}_\varepsilon(z)-\widetilde{B}_\varepsilon(z))$$

can also be extended to a bounded operator in $\mathcal{B}^0(\Sigma)$. We denote this extension also by $\overline{B}_\varepsilon(z)$ and according to (4.28), (B.3), (B.15) and (B.16) it has the representation

$$\overline{B}_\varepsilon(z)f(t)(x_\Sigma)=\int_{-1}^{1}\int_\Sigma G_z(x_\Sigma+\varepsilon(t-s)\nu(x_\Sigma)-y_\Sigma)f(s)(y_\Sigma)\,d\sigma(y_\Sigma)\,ds \tag{5.7}$$

for $f\in\mathcal{B}^0(\Sigma)$, a.e. $t\in(-1,1)$ and σ-a.e. $x_\Sigma\in\Sigma$. According to Lemma 4.7, (4.27) and (B.1) we have

$$\begin{aligned}\|\overline{B}_\varepsilon(z)-B_\varepsilon(z)\|_{0\to0}&\le\|\overline{B}_\varepsilon(z)-\widetilde{B}_\varepsilon(z)\|_{0\to0}+\|\widetilde{B}_\varepsilon(z)-B_\varepsilon(z)\|_{0\to0}\\&\le\|\overline{B}_\varepsilon(z)-\widetilde{B}_\varepsilon(z)\|_{0\to1/2}+\|B_\varepsilon(z)M_\varepsilon^{-1}(I-M_\varepsilon)\|_{0\to0}\\&\le C\varepsilon^{1/2}(1+|\log(\varepsilon)|)^{1/2}\qquad\forall\varepsilon\in(0,\varepsilon_{ABC})\end{aligned} \tag{5.8}$$

with ε_{ABC} defined by (4.19). This leads us to studying $\overline{B}_\varepsilon(z)$ instead of $B_\varepsilon(z)$. Next, we transfer this operator, which acts in $\mathcal{B}^0(\Sigma)$, to $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. To do so, let $\zeta \in C_b^2(\mathbb{R}^\theta; \mathbb{R})$, $\kappa \in \mathrm{SO}(\theta)$ and

$$\Sigma = \Sigma_{\zeta,\kappa} := \{\kappa(x', \zeta(x')) : x' \in \mathbb{R}^{\theta-1}\},$$

where we used the convention from Section 2.1 (xvi) that $(x', \zeta(x'))$ is an abbreviation for $(x'^T, \zeta(x'))^T$. Then, we introduce the isomorphism

$$\iota_{\zeta,\kappa} : L^2(\Sigma_{\zeta,\kappa}; \mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N), \quad (\iota_{\zeta,\kappa} f)(x') := f\big(\kappa(x', \zeta(x'))\big). \tag{5.9}$$

Through transforming integrals on $\Sigma_{\zeta,\kappa}$ to integrals on $\mathbb{R}^{\theta-1}$ we get the following norms for $\iota_{\zeta,\kappa}$ and its inverse:

$$\begin{aligned} \|\iota_{\zeta,\kappa}\|_{L^2(\Sigma_{\zeta,\kappa};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &= \sup_{x' \in \mathbb{R}^{\theta-1}} (1 + |\nabla\zeta(x')|^2)^{-1/4}, \\ \big\|\iota_{\zeta,\kappa}^{-1}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\Sigma_{\zeta,\kappa};\mathbb{C}^N)} &= \sup_{x' \in \mathbb{R}^{\theta-1}} (1 + |\nabla\zeta(x')|^2)^{1/4}. \end{aligned} \tag{5.10}$$

Note that the definition of $H^r(\Sigma_{\zeta,\kappa}; \mathbb{C}^N)$, $r \in [0, 2]$, see (2.2), implies that $\iota_{\zeta,\kappa}$ also acts as an isomorphic operator from $H^r(\Sigma_{\zeta,\kappa}; \mathbb{C}^N)$ to $H^r(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ for $r \in [0, 2]$. Recall that in this case $\iota_{\zeta,\kappa}$ can also be viewed as a bounded operator from $\mathcal{B}^r(\Sigma_{\zeta,\kappa})$ to $\mathcal{B}^r(\mathbb{R}^{\theta-1})$ which has the same norm as the operator acting from $H^r(\Sigma_{\zeta,\kappa}; \mathbb{C}^N)$ to $H^r(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$; cf. (2.9).

In the upcoming lines we often use $\Sigma_{\zeta,\kappa}$ in the upper index of various already introduced objects which depended on Σ. In this way we emphasize that the object with the upper index depends on ζ and κ.

We introduce for $\varepsilon \in (0, \varepsilon_{ABC}^{\Sigma_{\zeta,\kappa}})$ the operators

$$\begin{aligned} D_\varepsilon^{\zeta,\kappa}(z) &:= \iota_{\zeta,\kappa} \overline{B}_\varepsilon^{\Sigma_{\zeta,\kappa}}(z) \iota_{\zeta,\kappa}^{-1} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \\ D_0^{\zeta,\kappa}(z) &:= \iota_{\zeta,\kappa} B_0^{\Sigma_{\zeta,\kappa}}(z) \iota_{\zeta,\kappa}^{-1} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}). \end{aligned} \tag{5.11}$$

The results from Proposition 4.10 and (5.8) imply that $D_\varepsilon^{\zeta,\kappa}(z)$ is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_{ABC}^{\Sigma_{\zeta,\kappa}})$ and for $r \in (0, 1/2)$ the inequality

$$\|D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z)\|_{1/2\to 0} = \big\|\iota_{\zeta,\kappa}(B_0^{\Sigma_{\zeta,\kappa}}(z) - \overline{B}_\varepsilon^{\Sigma_{\zeta,\kappa}}(z))\iota_{\zeta,\kappa}^{-1}\big\|_{1/2\to 0} \le C\varepsilon^{1/2-r} \tag{5.12}$$

holds for all $\varepsilon \in (0, \varepsilon_{ABC}^{\Sigma_{\zeta,\kappa}})$. In particular, $D_\varepsilon^{\zeta,\kappa}(z) f$ converges as $\varepsilon \to 0$ to $D_0^{\zeta,\kappa}(z) f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ for $f \in \mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$. Furthermore, $\mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$ is by Proposition 2.21 (iii) a dense subset of $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Combining these considerations with the uniform boundedness of $D_\varepsilon^{\zeta,\kappa}(z)$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ shows that for all $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$

$$D_\varepsilon^{\zeta,\kappa}(z) f \xrightarrow{\varepsilon\to 0} D_0^{\zeta,\kappa}(z) f \quad \text{in } \mathcal{B}^0(\mathbb{R}^{\theta-1}). \tag{5.13}$$

Using (5.7) and (5.9), and setting

$$\varkappa_{\zeta,\kappa} = \kappa(\cdot, \zeta(\cdot)) \quad \text{and} \quad \nu_{\zeta,\kappa} = \nu^{\Sigma_{\zeta,\kappa}} \circ \varkappa_{\zeta,\kappa} = \frac{\kappa(-\nabla\zeta, 1)}{\sqrt{1 + |\nabla\zeta|^2}} \tag{5.14}$$

yields for $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ and a.e. $(t, x') \in (-1, 1) \times \mathbb{R}^{\theta-1}$

$$\begin{aligned} D_\varepsilon^{\zeta,\kappa}(z)f(t)(x') = \int_{-1}^{1} \int_{\mathbb{R}^{\theta-1}} & G_z\big(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \varepsilon(t-s)\nu_{\zeta,\kappa}(x')\big) \\ & \cdot \sqrt{1 + |\nabla\zeta(y')|^2} f(s)(y')\, dy'\, ds. \end{aligned} \tag{5.15}$$

Another useful representation is given by

$$D_\varepsilon^{\zeta,\kappa}(z)f(t) = \int_{-1}^{1} d_{\varepsilon(t-s)}^{\zeta,\kappa}(z) f(s)\, ds, \quad f \in \mathcal{B}^0(\mathbb{R}^{\theta-1}), t \in (-1, 1), \tag{5.16}$$

with the operator

$$\begin{aligned} & d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) : L^2(\Sigma; \mathbb{C}^N) \to L^2(\Sigma; \mathbb{C}^N), \\ d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)g(x') = \int_{\mathbb{R}^{\theta-1}} & G_z\big(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')\big) \\ & \cdot \sqrt{1 + |\nabla\zeta(y')|^2} g(y')\, dy', \end{aligned} \tag{5.17}$$

for $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}^{\Sigma_{\zeta,\kappa}}, 2\varepsilon_{ABC}^{\Sigma_{\zeta,\kappa}}) \setminus \{0\}$. For the interaction strengths $\eta^{\Sigma_{\zeta,\kappa}}, \tau^{\Sigma_{\zeta,\kappa}} \in C_b^1(\Sigma_{\zeta,\kappa}; \mathbb{R})$ we also define the matrix-valued function

$$Q_{\eta,\tau}^{\zeta,\kappa} := V^{\Sigma_{\zeta,\kappa}} \circ \varkappa_{\zeta,\kappa} = \eta^{\Sigma_{\zeta,\kappa}} \circ \varkappa_{\zeta,\kappa} I_N + \tau^{\Sigma_{\zeta,\kappa}} \circ \varkappa_{\zeta,\kappa}\beta. \tag{5.18}$$

There holds $Q_{\eta,\tau}^{\zeta,\kappa} = \iota_{\zeta,\kappa} V^{\Sigma_{\zeta,\kappa}} \iota_{\zeta,\kappa}^{-1}$ in the sense of operators in $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$.

5.1.1 Hyperplanes and constant interaction strengths

In this section we assume that $\Sigma = \Sigma_{y_0,\kappa}$, for a $y_0 \in \mathbb{R}$ and a $\kappa \in \mathrm{SO}(\theta)$, i.e. Σ is an affine $(\theta - 1)$-dimensional hyperplane in $\mathbb{R}^\theta$. We also assume that the interaction strengths are constant and given by $\eta, \tau \in \mathbb{R}$, i.e. $\eta^{\Sigma_{y_0,\kappa}} \equiv \eta \in \mathbb{R}$ and $\tau^{\Sigma_{y_0,\kappa}} \equiv \tau \in \mathbb{R}$. This implies that $Q_{\eta,\tau}^{y_0,\kappa}$ is equal to the constant matrix

$$Q_{\eta,\tau} := \eta I_N + \tau\beta \tag{5.19}$$

in this case. The main goal of this section is to show that for every compact set $S \subset \mathbb{R}^2$ satisfying

$$\max_{(\eta,\tau)\in S} \eta^2 - \tau^2 < \frac{\pi^2}{4},$$

there exists an $\varepsilon_{\mathrm{pl},2} = \varepsilon_{\mathrm{pl},2}(S)$ such that $\|(I + D_\varepsilon^{y_0,\kappa}(z)Q_{\eta,\tau})^{-1}\|_{0\to 0}$ is uniformly bounded with respect to $(\varepsilon, y_0, (\eta,\tau), \kappa) \in (0, \varepsilon_{\mathrm{pl},2}) \times \mathbb{R} \times S \times \mathrm{SO}(\theta)$; cf. Corollary 5.9. This result plays a major role when we prove the uniform boundedness of the operators $(I + B_\varepsilon(z)Vq)^{-1}$ in the case that Σ is a rotated C_b^2-graph in Section 5.1.2.

We proceed in this section as follows: First, we use the Fourier transform to turn $D_\varepsilon^{y_0,\kappa}(z)$ into a decomposable operator with frequency dependent fiber operators; see Lemma 5.1–Definition 5.3. Then, up to Lemma 5.7, we find and analyse suitable approximations for the fiber operators for high and low frequencies. Finally, we use these results to prove the main statements (Proposition 5.8 and Corollary 5.9) of this section.

Before we start, let us fix some notations. In the current setting the normal vector $\nu_{\zeta,\kappa}$ is constant and given by κe_θ, where e_θ is the θ-th Euclidean unit vector. Moreover, the map $\iota^{\Sigma_{y_0,\kappa}}$, see Definition 2.7, is a bijective isomorphism and the Weingarten map $W^{\Sigma_{y_0,\kappa}}$ is zero in this case. Hence, by revisiting Proposition 2.12 we can set $\varepsilon_{\mathrm{tub}}^{\Sigma_{y_0,\kappa}}$ to ∞. Thus, (4.19) and the constancy of the unit-vector (and its constant extension to $\mathbb{R}^\theta$) lets us also set $\varepsilon_{ABC}^{\Sigma_{y_0,\kappa}}$ to infinity. According to (5.14) and (5.15) $D_\varepsilon^{y_0,\kappa}(z)$ has for $\varepsilon \in (0,\infty)$ and $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ the representation

$$D_\varepsilon^{y_0,\kappa}(z)f(t)(x') = \int_{-1}^{1}\int_{\mathbb{R}^{\theta-1}} G_z\big(\kappa(x'-y', \varepsilon(t-s))\big) f(s)(y')\, dy'\, ds \tag{5.20}$$

for a.e. $(t,x') \in (-1,1) \times \mathbb{R}^{\theta-1}$, which shows that $D_\varepsilon^{y_0,\kappa}(z)$ is independent of $y_0 \in \mathbb{R}$. Furthermore, (5.13) implies that also $D_0^{y_0,\kappa}(z)$ is independent of y_0. Thus, w.l.o.g. we can set $y_0 = 0$. We define the matrices

$$\widetilde{\alpha}_j := \alpha \cdot \kappa e_j,\ j \in \{1,\dots,\theta\}, \quad \text{and} \quad \widetilde{\alpha}\cdot\xi := \sum_{j=1}^{\theta} \widetilde{\alpha}_j \xi_j,\ \xi \in \mathbb{R}^\theta, \tag{5.21}$$

as well as

$$\widetilde{\alpha}'\cdot\xi' = \sum_{j=1}^{\theta-1} \widetilde{\alpha}_j \xi_j', \qquad \xi' \in \mathbb{R}^{\theta-1}, \tag{5.22}$$

for convenience. Similarly to the α-matrices from Definition 3.1, the $\widetilde{\alpha}$-matrices are self-adjoint, unitary and fulfil the relations

$$\widetilde{\alpha}_j\widetilde{\alpha}_l + \widetilde{\alpha}_l\widetilde{\alpha}_j = 2\delta_{jl} \quad \text{and} \quad \widetilde{\alpha}_j\beta + \beta\widetilde{\alpha}_j = 0 \quad \forall j,l \in \{1,\dots,\theta\}. \tag{5.23}$$

Using these rules one easily concludes

$$(\widetilde{\alpha}\cdot\xi)^2 = |\xi|^2 I_N \quad \text{and} \quad (\widetilde{\alpha}'\cdot\xi')^2 = |\xi'|^2 I_N \quad \forall \xi = (\xi',\xi_\theta) \in \mathbb{R}^\theta. \tag{5.24}$$

We start by calculating the Fourier transform of the function $G_z(\kappa(\cdot,\widetilde{\varepsilon}))$ for fixed $\widetilde{\varepsilon} \neq 0$; cf. [60, eqs. (44)–(45)] for similar considerations.

Lemma 5.1. *Let $z \in \rho(H)$, G_z be the integral kernel of $(H-z)^{-1}$ given by* (3.3)–(3.4)*, $\widetilde{\varepsilon} \neq 0$ and $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^{\theta-1}$ from Section 2.1* (xvii)*. Then,*

$$\mathcal{F}G_z(\kappa(\cdot,\widetilde{\varepsilon})) = \left(\frac{\widetilde{\alpha}'\cdot(\cdot) + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\cdot|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(\widetilde{\varepsilon})\right)\frac{ie^{|\widetilde{\varepsilon}|i\sqrt{z^2-m^2-|\cdot|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}.$$

Proof. Let $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_{1,2}$ be as defined in Section 2.1 (xvii). We start by considering $\mathcal{F}_1 G_z(\kappa(\cdot))$. Since $G_z(\kappa(\cdot)) \in L^1(\mathbb{R}^\theta;\mathbb{C}^{N\times N}) \subset \mathcal{S}'(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$, see Proposition 3.4, the expression $\mathcal{F}_1 G_z(\kappa(\cdot))$ is well-defined in $\mathcal{S}'(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$. Moreover, $\mathcal{F}_1 G_z(\kappa(\cdot)) = \mathcal{F}_2^{-1}\mathcal{F}_{1,2} G_z(\kappa(\cdot))$. Thus, we calculate $\mathcal{F}_{1,2}G_z(\kappa(\cdot))$ next. The function G_z satisfies the equation

$$(-i(\alpha\cdot\nabla) + m\beta - zI_N)G_z = \delta I_N,$$

with δ denoting the δ-distribution supported in $\{0\}$. Hence, the standard rules for the Fourier transform, see [63, Chapter IX], show

$$\big(\alpha\cdot(\cdot) + m\beta - zI_N\big)\mathcal{F}_{1,2}G_z = \frac{1}{\sqrt{(2\pi)^\theta}} I_N \quad \text{in } \mathcal{S}'(\mathbb{R}^\theta;\mathbb{C}^{N\times N}).$$

Furthermore, $G_z \in L^1(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$ implies $\mathcal{F}_{1,2}G_z \in C_0(\mathbb{R}^\theta;\mathbb{C}^{N\times N})$; see [63, Theorem IX.7]. Thus, using the properties of α_j, $j\in\{1,\dots,\theta\}$, and β yields

$$\mathcal{F}_{1,2}G_z(\xi) = \frac{1}{\sqrt{(2\pi)^\theta}}(\alpha\cdot\xi + m\beta - zI_N)^{-1} = \frac{\alpha\cdot\xi + m\beta + zI_N}{(|\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}} \quad \forall \xi\in\mathbb{R}^\theta.$$

Consequently,

$$(\mathcal{F}_{1,2}G_z)(\kappa\xi) = \frac{\alpha\cdot(\kappa\xi) + m\beta + zI_N}{(|\kappa\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}} \quad \forall \xi\in\mathbb{R}^\theta.$$

Additionally, $\kappa\in\mathrm{SO}(\theta)$ gives us

$$\mathcal{F}_{1,2}G_z(\kappa(\cdot))(\xi) = (\mathcal{F}_{1,2}G_z)(\kappa\xi) = \frac{\widetilde{\alpha}\cdot\xi + m\beta + zI_N}{(|\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}} \quad \forall \xi\in\mathbb{R}^\theta. \tag{5.25}$$

Next, we determine $\mathcal{F}_2^{-1}\mathcal{F}_{1,2}G_z$. We claim that for a.e. $(\xi', x_\theta)\in\mathbb{R}^\theta$ the equation

$$\begin{aligned}\mathcal{F}_2^{-1}&\mathcal{F}_{1,2}G_z(\kappa(\cdot))(\xi', x_\theta)\\ &= \left(\frac{\widetilde{\alpha}'\cdot\xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta\mathrm{sign}(x_\theta)\right)\frac{ie^{|x_\theta| i\sqrt{z^2-m^2-|\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}\end{aligned} \tag{5.26}$$

holds. We verify (5.26) by applying $\mathcal{F}_2$ and comparing the result with (5.25). As the right-hand side of (5.26) decays exponentially for $|x_\theta|\to\infty$, we can use the integral representation of the Fourier transform. Hence, simple integration gives us for $\xi = (\xi', \xi_\theta) \in \mathbb{R}^\theta$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(x_\theta) \right) \cdot \frac{i e^{i(-x_\theta \xi_\theta + |x_\theta| \sqrt{z^2 - m^2 - |\xi'|^2})}}{2\sqrt{(2\pi)^{\theta-1}}} dx_\theta = \frac{\widetilde{\alpha} \cdot \xi + m\beta + zI_N}{(|\xi|^2 + m^2 - z^2)\sqrt{(2\pi)^\theta}},$$

which verifies (5.26). Therefore, $\mathcal{F}_1 G_z(\kappa(\cdot)) = \mathcal{F}_2^{-1} \mathcal{F}_{1,2} G_z(\kappa(\cdot))$ can be represented by the function

$$\mathbb{R}^\theta \ni (\xi', x_\theta) \mapsto \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(x_\theta) \right) \frac{i e^{|x_\theta| i \sqrt{z^2 - m^2 - |\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}}.$$

Moreover, since $G_z(\kappa(\cdot, \widetilde{\varepsilon})) \in L^1(\mathbb{R}^{\theta-1}; \mathbb{C}^{N\times N})$ for $\widetilde{\varepsilon} \neq 0$, which follows from Proposition 3.4, this shows that for $\widetilde{\varepsilon} \neq 0$ and $\xi' \in \mathbb{R}^{\theta-1}$ the equation

$$\begin{aligned} \mathcal{F} G_z(\kappa(\cdot, \widetilde{\varepsilon}))(\xi') &= \frac{1}{\sqrt{(2\pi)^{\theta-1}}} \int_{\mathbb{R}^{\theta-1}} G_z(\kappa(x', \widetilde{\varepsilon})) e^{-i\langle x', \xi' \rangle} \, dx' \\ &= \mathcal{F}_1 G_z(\kappa(\cdot))(\xi', \widetilde{\varepsilon}) \\ &= \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(\widetilde{\varepsilon}) \right) \frac{i e^{|\widetilde{\varepsilon}| i \sqrt{z^2 - m^2 - |\xi'|^2}}}{2\sqrt{(2\pi)^{\theta-1}}} \end{aligned}$$

holds. □

Proposition 5.2. *Let $z \in \rho(H)$, $\varepsilon > 0$ and $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^{\theta-1}$ from Section 2.1* (xvii). *Then, for $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$*

$$\mathcal{F} D_\varepsilon^{0,\kappa}(z) \mathcal{F}^{-1} f(t)(\xi') = \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(t - s) \right) \cdot \frac{i e^{|\varepsilon(t-s)| i \sqrt{z^2 - m^2 - |\xi'|^2}}}{2} f(s)(\xi') \, ds \tag{5.27}$$

and

$$\mathcal{F} D_0^{0,\kappa}(z) \mathcal{F}^{-1} f(t)(\xi') = \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \mathrm{sign}(t - s) \right) \cdot \frac{i f(s)(\xi')}{2} \, ds \tag{5.28}$$

for a.e $(t, \xi') \in (-1, 1) \times \mathbb{R}^{\theta-1}$.

Proof. We start with the case $\varepsilon > 0$. Using (5.20) shows

$$D_\varepsilon^{0,\kappa}(z)f(t) = \int_{-1}^{1} G_z(\kappa(\cdot, \varepsilon(t-s))) * f(s)\, ds$$

for a.e. $t \in (-1,1)$ and all $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$. Thus, Lemma 5.1 and [63, Theorem IX.4] prove the statement for $\varepsilon > 0$. It remains to consider the operator $D_0^{0,\kappa}(z)$. We start by defining

$$\widetilde{D}_0^{0,\kappa}(z) : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}),$$
$$\widetilde{D}_0^{0,\kappa}(z)f(t)(\xi') := \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \right) \frac{if(s)(\xi')}{2}\, ds.$$

Next, let $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$. From (5.27) and the dominated convergence theorem, see Proposition 2.16, one obtains that $\mathcal{F}D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}f$ converges for $\varepsilon \to 0$ to $\widetilde{D}_0^{0,\kappa}(z)f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Thus, the boundedness of $\mathcal{F}$ and $\mathcal{F}^{-1}$ in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ (and therefore also in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, cf. (2.9)), implies that $D_\varepsilon^{0,\kappa}(z)f$ converges to $\mathcal{F}^{-1}\widetilde{D}_0^{0,\kappa}(z)\mathcal{F}f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, by (5.13) $D_\varepsilon^{0,\kappa}(z)f$ converges to $D_0^{0,\kappa}(z)f$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Hence, $D_0^{0,\kappa}(z)f = \mathcal{F}^{-1}\widetilde{D}_0^{0,\kappa}(z)\mathcal{F}f$ which proves (5.28). ☐

The structure of $\mathcal{F}D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}$ and $\mathcal{F}D_0^{0,\kappa}(z)\mathcal{F}^{-1}$ inspires us to change our viewpoint. Namely, instead of viewing these operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ we consider them as operators in the isometrically isomorphic space $L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N))$. In the context of direct integrals the notation $\int_{\mathbb{R}^{\theta-1}}^{\oplus} L^2((-1,1);\mathbb{C}^N))\, d\xi'$ for this space is also common. Considered as operators acting in this space $\mathcal{F}D_\varepsilon^{0,\kappa}(z)\mathcal{F}^{-1}$ and $\mathcal{F}D_0^{0,\kappa}(z)\mathcal{F}^{-1}$ are decomposable operators in $L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N))$ with the following fibers.

Definition 5.3. *Let $\varepsilon > 0$, $\xi' \in \mathbb{R}^{\theta-1}$ and $z \in \rho(H)$. We define*

$$\mathfrak{D}_{\varepsilon,\xi'}(z) : L^2((-1,1);\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N),$$
$$\mathfrak{D}_{\varepsilon,\xi'}(z)f(t) := \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \right) \cdot \frac{ie^{|\varepsilon(t-s)|i\sqrt{z^2-m^2-|\xi'|^2}}}{2} f(s)\, ds,$$

and

$$\mathfrak{D}_{0,\xi'}(z) : L^2((-1,1);\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N),$$
$$\mathfrak{D}_{0,\xi'}(z)f(t) := \int_{-1}^{1} \left(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \right) \frac{if(s)}{2}\, ds.$$

Remark 5.4. *These operators still depend on the rotation matrix $\kappa \in \mathrm{SO}(\theta)$ since the $\widetilde{\alpha}$'s depend on κ. As we use these operators only as auxiliary operators in this section, we omit κ.*

Next, we explain the ideas mentioned above Definition 5.3 in a more rigorous way. Using

$$\begin{aligned}\mathcal{B}^0(\mathbb{R}^{\theta-1}) &= L^2((-1,1); L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N))\\ &\simeq L^2((-1,1)\times\mathbb{R}^{\theta-1};\mathbb{C}^N) \simeq L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)),\end{aligned}$$

see Proposition 2.18 (iii), allows us to define the isometric isomorphism

$$\begin{aligned}\mathfrak{i} : \mathcal{B}^0(\mathbb{R}^{\theta-1}) &\to L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)),\\ \mathfrak{i}f(\xi')(t) &:= \mathcal{F}f(t)(\xi') \quad \text{for a.e. } (\xi',t)\in\mathbb{R}^{\theta-1}\times(-1,1).\end{aligned}$$

Thus, by Proposition 5.2 and Definition 5.3 we obtain for $\varepsilon \geq 0$

$$\begin{aligned}\mathfrak{i}D_\varepsilon^{0,\kappa}(z)\mathfrak{i}^{-1} : L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)) &\to L^2(\mathbb{R}^{\theta-1}; L^2((-1,1);\mathbb{C}^N)),\\ \mathfrak{i}D_\varepsilon^{0,\kappa}(z)\mathfrak{i}^{-1}f(\xi') &= \mathfrak{D}_{\varepsilon,\xi'}(z)f(\xi'),\end{aligned} \tag{5.29}$$

i.e. $\mathfrak{i}D_\varepsilon^{0,\kappa}(z)\mathfrak{i}^{-1}$ is a decomposable operator in the sense of (2.12) which is induced by the operator-valued function $\zeta' \mapsto \mathfrak{D}_{\varepsilon,\xi'}(z)$. Hence, Proposition 2.19 lets us transfer results regarding $\mathfrak{D}_{\varepsilon,\xi'}(z)$ to $D_\varepsilon^{0,\kappa}(z)$ and vice versa.

Next, we study the operator $\mathfrak{D}_{\varepsilon,\xi'}(z)$ in detail. For this purpose, we introduce the auxiliary operator

$$\begin{aligned}\mathfrak{H}_{\rho,w'} &: L^2((-1,1);\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N),\\ (\mathfrak{H}_{\rho,w'}f)(t) &:= \int_{-1}^{1}\big(\widetilde{\alpha}'\cdot w' + i\widetilde{\alpha}_\theta \operatorname{sign}(t-s)\big)\frac{e^{-\rho|t-s|}}{2}f(s)\,ds,\end{aligned} \tag{5.30}$$

for $\rho\in[0,\infty)$ and $w'\in\mathbb{R}^{\theta-1}$ with $|w'|=1$. It is easy to check that $\mathfrak{H}_{\rho,w'}$ is a self-adjoint Hilbert-Schmidt operator.

Lemma 5.5. *Let $\varepsilon > 0$, $\xi'\in\mathbb{R}^{\theta-1}\setminus\{0\}$ and $z\in\rho(H)$. Then, there exists a constant $C_{\mathrm{pl},1} > 0$ which only depends on m and z such that*

$$\begin{aligned}\|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} &\leq C_{\mathrm{pl},1}\varepsilon(1+|\xi'|),\\ \|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} &\leq \frac{C_{\mathrm{pl},1}}{1+|\xi'|}.\end{aligned}$$

Proof. In this proof $C>0$ denotes a constant which may change in-between lines, but only depends on m and z. We start by estimating the kernel of $\mathfrak{D}_{\varepsilon,\xi'}(z)-\mathfrak{D}_{0,\xi'}(z)$. We can bound this kernel by

$$\begin{aligned}C\Big|1-e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\Big| &\leq C\varepsilon|t-s|\sqrt{|z^2-m^2-|\xi'|^2|}\\ &\leq C\varepsilon(1+|\xi'|).\end{aligned}$$

Hence, there exists a constant $C_{\mathrm{pl},1} > 0$ such that

$$\|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \leq C_{\mathrm{pl},1}\varepsilon(1+|\xi'|).$$

Next, we estimate the kernel of $\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}$ by

$$\begin{aligned}\frac{1}{2}\bigg|&\Big(e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\Big)\Big(\widetilde{\alpha}'\cdot\frac{\xi'}{|\xi'|} + i\operatorname{sign}(t-s)\widetilde{\alpha}_\theta\Big)\bigg| \\ &+ \frac{1}{2}\bigg|e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\Big(\widetilde{\alpha}'\cdot\frac{\xi'}{|\xi'|}\Big)\bigg(1-\frac{i|\xi'|}{\sqrt{z^2-m^2-|\xi'|^2}}\bigg)\bigg| \\ &+ \frac{1}{2}\bigg|e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\frac{m\beta + zI_N}{\sqrt{z^2-m^2-|\xi'|^2}}\bigg|.\end{aligned}\tag{5.31}$$

The first term in (5.31) can be estimated for $\varepsilon|t-s| \leq 1$ by

$$\begin{aligned}C\Big|e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\Big| &\leq C\varepsilon|t-s|\Big||\xi'| + i\sqrt{z^2-m^2-|\xi'|^2}\Big|. \\ &\leq C\frac{|z^2-m^2|}{\big||\xi'| - i\sqrt{z^2-m^2-|\xi'|^2}\big|} \\ &\leq \frac{C}{1+|\xi'|},\end{aligned}\tag{5.32}$$

where we used $z \in \rho(H) = \mathbb{C}\setminus\big((-\infty,-|m|]\cup[|m|,\infty)\big)$ as well as $\operatorname{Im}\sqrt{w} > 0$ for $w \in \mathbb{C}\setminus[0,\infty)$. For $\varepsilon|t-s| > 1$, we get

$$C\Big|e^{-\varepsilon|t-s||\xi'|} - e^{\varepsilon|t-s|i\sqrt{z^2-m^2-|\xi'|^2}}\Big| \leq C\big(e^{-|\xi'|} + e^{-\operatorname{Im}\sqrt{z^2-m^2-|\xi'|^2}}\big) \leq \frac{C}{1+|\xi'|}.\tag{5.33}$$

Similarly as we estimated the first term in the case $\varepsilon|t-s| \leq 1$, the second term in (5.31) can be bounded by

$$\begin{aligned}C\bigg|1-\frac{i|\xi'|}{\sqrt{z^2-m^2-|\xi'|^2}}\bigg| &= C\frac{|z^2-m^2|}{\Big|\sqrt{z^2-m^2-|\xi'|^2}+i|\xi'|\Big|\Big|\sqrt{z^2-m^2-|\xi'|^2}\Big|} \\ &\leq \frac{C}{(1+|\xi'|)^2} \leq \frac{C}{1+|\xi'|}.\end{aligned}$$

One also sees that the third term in (5.31) is smaller than $\frac{C}{1+|\xi'|}$ for a sufficiently large C. Summing up, we have that the kernel of $\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}$ can be bounded by $\frac{C}{1+|\xi'|}$ and therefore if $C_{\mathrm{pl},1}$ is chosen sufficiently large, then

$$\|\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \leq \frac{C_{\mathrm{pl},1}}{1+|\xi'|}.$$ □

Lemma 5.6. *Let $\rho \geq 0$, $w' \in \mathbb{R}^{\theta-1}$ with $|w'| = 1$ and q be as in (5.1). Then, $\sigma(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}) \subset [-\frac{2}{\pi}, \frac{2}{\pi}]$.*

Proof. To shorten notation, we set $\widetilde{\alpha}_\pm := \widetilde{\alpha}' \cdot w' \pm i\widetilde{\alpha}_\theta$. Then,

$$\mathfrak{H}_{\rho,w'}f(t) = \frac{1}{2}\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_- f(s)\,ds + \frac{1}{2}\int_{-1}^t e^{-\rho|t-s|}\widetilde{\alpha}_+ f(s)\,ds$$

for $f \in L^2((-1,1);\mathbb{C}^N)$. Using the rules for the $\widetilde{\alpha}$ matrices from (5.23) yields the orthogonality relation $\operatorname{ran}\widetilde{\alpha}_+ \perp \operatorname{ran}\widetilde{\alpha}_-$. Hence, as $q \geq 0$ a.e. on $(-1,1)$ by (5.1), we have for $f \in L^2((-1,1);\mathbb{C}^N)$

$$\begin{aligned}
&\big\|\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}f\big\|^2_{L^2((-1,1);\mathbb{C}^N)} \\
&\quad=\frac{1}{4}\int_{-1}^1 q(t)\Big|\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\Big|^2\,dt \\
&\qquad+\frac{1}{4}\int_{-1}^1 q(t)\Big|\int_{-1}^t e^{-\rho|t-s|}\widetilde{\alpha}_+\sqrt{q(s)}f(s)\,ds\Big|^2\,dt.
\end{aligned}\tag{5.34}$$

We start by estimating the first term on the right-hand side. We define the function $\mathscr{Q}(t) = -1 + \int_{-1}^t q(s)\,ds, t \in [-1,1]$. Then, $\mathscr{Q}' = q$, $\mathscr{Q}(-1) = -1$ and $\mathscr{Q}(1) = 0$ since $\int_{-1}^1 q(s)\,ds = 1$. Applying the Cauchy-Schwarz inequality and Fubini's theorem gives us

$$\begin{aligned}
&\frac{1}{4}\int_{-1}^1 q(t)\Big|\int_t^1 e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\Big|^2\,dt \\
&\quad= \frac{1}{4}\int_{-1}^1 q(t)\Big|\int_t^1 \frac{\sqrt{\cos\big(\frac{\pi}{2}\mathscr{Q}(s)\big)}}{\sqrt{\cos\big(\frac{\pi}{2}\mathscr{Q}(s)\big)}}e^{-\rho|t-s|}\widetilde{\alpha}_-\sqrt{q(s)}f(s)\,ds\Big|^2\,dt \\
&\quad\leq \frac{1}{4}\int_{-1}^1 q(t)\bigg(\int_t^1 \cos\big(\tfrac{\pi}{2}\mathscr{Q}(s)\big)q(s)\,ds\bigg)\bigg(\int_t^1 \frac{1}{\cos\big(\frac{\pi}{2}\mathscr{Q}(s)\big)}|\widetilde{\alpha}_- f(s)|^2\,ds\bigg)\,dt \\
&\quad= \frac{1}{2\pi}\int_{-1}^1 -\sin\big(\tfrac{\pi}{2}\mathscr{Q}(t)\big)q(t)\bigg(\int_t^1 \frac{1}{\cos\big(\frac{\pi}{2}\mathscr{Q}(s)\big)}|\widetilde{\alpha}_- f(s)|^2\,ds\bigg)\,dt \\
&\quad= \frac{1}{2\pi}\int_{-1}^1 \bigg(\int_{-1}^s -\sin\big(\tfrac{\pi}{2}\mathscr{Q}(t)\big)q(t)\,dt\bigg)\frac{1}{\cos\big(\frac{\pi}{2}\mathscr{Q}(s)\big)}|\widetilde{\alpha}_- f(s)|^2\,ds \\
&\quad= \frac{1}{\pi^2}\int_{-1}^1 |\widetilde{\alpha}_- f(s)|^2\,ds.
\end{aligned}$$

The same trick with $\mathscr{Q}+1$ instead of $\mathscr{Q}$ yields that the second term of the right-hand side of equation (5.34) can be estimated by $\frac{1}{\pi^2}\int_{-1}^1 |\widetilde{\alpha}_+ f(s)|^2\,ds$. These estimates,

$\operatorname{ran}\widetilde{\alpha}_+ \perp \operatorname{ran}\widetilde{\alpha}_-$ and (5.24) imply

$$\begin{aligned}
\big\|\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}f\big\|^2_{L^2((-1,1);\mathbb{C}^N)} &\leq \frac{1}{\pi^2}\int_{-1}^{1}|\widetilde{\alpha}_- f(s)|^2 + |\widetilde{\alpha}_+ f(s)|^2\,ds \\
&= \frac{1}{\pi^2}\int_{-1}^{1}|(\widetilde{\alpha}_- + \widetilde{\alpha}_+)f(s)|^2\,ds \\
&= \frac{1}{\pi^2}\int_{-1}^{1}|2(\widetilde{\alpha}'\cdot w')f(s)|^2\,ds \\
&= \frac{4}{\pi^2}\|f\|^2_{L^2((-1,1);\mathbb{C}^N)}.
\end{aligned}$$

Since $\mathfrak{H}_{\rho,w'}$ is self-adjoint in $L^2((-1,1);\mathbb{C}^N)$, we obtain $\sigma(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}) \subset [-\frac{2}{\pi},\frac{2}{\pi}]$. □

Having studied the spectrum of $\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q}$, we employ this knowledge to study the bounded invertibility of $I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q$. Recall that $Q_{\eta,\tau} = \eta I_N + \tau\beta$ for $\eta,\tau \in \mathbb{R}$.

Lemma 5.7. *Let $\rho \geq 0$, $w' \in \mathbb{R}^{\theta-1}$ with $|w'| = 1$, $\eta,\tau \in \mathbb{R}$, $Q_{\eta,\tau} = \eta I_N + \tau\beta$, $d = \eta^2 - \tau^2$,*

$$c(d) := \begin{cases} \sqrt{d}\frac{2}{\pi}, & d \geq 0, \\ 0, & d < 0, \end{cases}$$

and q be as in (5.1). If $d < \frac{\pi^2}{4}$, then $c(d) < 1$, $I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q$ is continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and the norm of the inverse is bounded by the constant

$$C_{\mathrm{pl},2} = C_{\mathrm{pl},2}(\eta,\tau) := 4\|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{(1 - c(d))\pi} + 1. \tag{5.35}$$

Proof. The identities $\mathfrak{H}_{\rho,w'}Q_{\eta,\tau} = Q_{\eta,-\tau}\mathfrak{H}_{\rho,w'}$ and $Q_{\eta,-\tau}Q_{\eta,\tau} = dI_N$, which follow from (5.23), give us

$$I - d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2 = (I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})(I - \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}).$$

If $d < \frac{\pi^2}{4}$, then Lemma 5.6 implies $1 \in \rho(d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2)$ and therefore the operator $I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}$ is also continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and

$$\begin{aligned}
&\big\|(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad = \big\|(I - d(\sqrt{q}\mathfrak{H}_{\rho,w'}\sqrt{q})^2)^{-1}(I - \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad \leq \frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{1 - c(d)}.
\end{aligned}$$

Moreover, applying Proposition 2.29 shows that $I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q$ is also continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and

$$\begin{aligned}
&\big\|(I + \mathfrak{H}_{\rho,w'}Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad = \big\|\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q}(I + \sqrt{q}\mathfrak{H}_{\rho,w'}Q_{\eta,\tau}\sqrt{q})^{-1}\sqrt{q} - I\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad \le \|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\|\mathfrak{H}_{\rho,w'}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{1 - c(d)} + 1 \\
&\quad \le 4\|q\|_{L^\infty((-1,1))}(|\eta| + |\tau|)\frac{1 + (|\eta| + |\tau|)\frac{2}{\pi}}{(1 - c(d))\pi} + 1,
\end{aligned}$$

where we used Lemma 5.6 (for $q = \frac{1}{2}$) to estimate $\|\mathfrak{H}_{\rho,w'}\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}$ by $\frac{4}{\pi}$. □

In the last part of Section 5.1.1 we use our findings to prove a norm estimate for the operator $(I + D_\varepsilon^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}$.

Proposition 5.8. *Let $\kappa \in \mathrm{SO}(\theta)$, $\eta, \tau \in \mathbb{R}$ fulfil $d = \eta^2 - \tau^2 < \frac{\pi^2}{4}$, $Q_{\eta,\tau} = \eta I_N + \tau\beta$, q be as in* (5.1) *and $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, let*

$$C_{\mathrm{pl},3} = C_{\mathrm{pl},3}(\eta, \tau, \kappa) := 2\max\{C_{\mathrm{pl},2}, \big\|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0}\} \tag{5.36}$$

with $C_{\mathrm{pl},2} = C_{\mathrm{pl},2}(\eta, \tau)$ from Lemma 5.7 and

$$\varepsilon_{\mathrm{pl},1} = \varepsilon_{\mathrm{pl},1}(\eta, \tau, \kappa) := (C_{\mathrm{pl},3}C_{\mathrm{pl},1}(|\eta| + |\tau|)\|q\|_{L^\infty((-1,1))})^{-2} \tag{5.37}$$

with $C_{\mathrm{pl},1}$ from Lemma 5.5. Then,

$$\sup_{\varepsilon\in(0,\varepsilon_{\mathrm{pl},1})} \big\|(I + D_\varepsilon^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0} \le C_{\mathrm{pl},3} < \infty.$$

Proof. We start by arguing that $C_{\mathrm{pl},3} < \infty$. The assumption $d < \frac{\pi^2}{4}$ guarantees $C_{\mathrm{pl},2} < \infty$. Moreover, applying Proposition 3.15 (iii) and Proposition 4.13 (for $V = Q_{\eta,\tau} = \eta I_2 + \tau\beta = \text{const.}$ and $r = 0$) shows that $I + B_0^{\Sigma_{y_0,\kappa}}(z)Q_{\eta,\tau}q$ is continuously invertible in $\mathcal{B}^0(\Sigma_{y_0,\kappa})$. Let us shortly explain why Proposition 4.13 is indeed applicable. We have to show $\cos\big(\frac{(\alpha\cdot\nu)Q_{\eta,\tau}}{2}\big)^{-1} \in W^1_\infty(\Sigma_{y_0,\kappa};\mathbb{C}^{N\times N})$ and that $I + \mathcal{C}_z^{\Sigma_{y_0,\kappa}}\widetilde{Q_{\eta,\tau}}$ is continuously invertible in $L^2(\Sigma_{y_0,\kappa};\mathbb{C}^N)$, where

$$\widetilde{Q_{\eta,\tau}} = Q_{\eta,\tau}\mathrm{sinc}\big(\tfrac{(\alpha\cdot\nu)Q_{\eta,\tau}}{2}\big)\cos\big(\tfrac{(\alpha\cdot\nu)Q_{\eta,\tau}}{2}\big)^{-1}.$$

In the same way as in (5.4)–(5.6) we get $\cos\big(\frac{(\alpha\cdot\nu)Q_{\eta,\tau}}{2}\big) = \cos\big(\frac{\sqrt{d}}{2}\big) = \text{const.}$ and $\widetilde{Q_{\eta,\tau}} = \widetilde{\eta}I_N + \widetilde{\tau}\beta$ with $(\widetilde{\eta}, \widetilde{\tau}) = \mathrm{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$. Hence, $d < \frac{\pi^2}{4}$ implies $\cos\big(\frac{\sqrt{d}}{2}\big) \neq 0$ and therefore $\cos\big(\frac{(\alpha\cdot\nu)Q_{\eta,\tau}}{2}\big)^{-1} \in W^1_\infty(\Sigma_{y_0,\kappa};\mathbb{C}^{N\times N})$. Furthermore, we have

$$\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4\tan\big(\tfrac{\sqrt{d}}{2}\big)^2 < 4$$

and thus Proposition 3.15 (iii) implies that $I + \mathcal{C}_z^{\Sigma_{y_0,\kappa}} \widetilde{Q_{\eta,\tau}}$ is continuously invertible in $L^2(\Sigma_{y_0,\kappa};\mathbb{C}^N)$. Hence, the assumptions of Proposition 4.13 are satisfied and its application is justified. By (5.11) $B_0^{\Sigma_{0,\kappa}}(z)$ is related to $D_0^{0,\kappa}(z)$ via an isometric isomorphism and therefore $I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, which proves $C_{\mathrm{pl},3} < \infty$.

According to Lemma 5.5 we have

$$\begin{aligned}
&\|\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\qquad \le C_{\mathrm{pl},1}\|q\|_{L^\infty((-1,1))}\frac{|\eta|+|\tau|}{1+|\xi'|} = \frac{\varepsilon_{\mathrm{pl},1}^{-1/2}}{C_{\mathrm{pl},3}(1+|\xi'|)}
\end{aligned}$$

for $\xi' \in \mathbb{R}^{\theta-1}\setminus\{0\}$ and all $\varepsilon > 0$. Hence, if we set $R := \varepsilon_{\mathrm{pl},1}^{-1/2} - 1$, the choices of $C_{\mathrm{pl},3}$, $\varepsilon_{\mathrm{pl},1}$ and R yield

$$\begin{aligned}
&\big\|(I+\mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad \cdot \big\|\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\quad \le C_{\mathrm{pl},2}\frac{\varepsilon_{\mathrm{pl},1}^{-1/2}}{C_{\mathrm{pl},3}(1+R)} \le \frac{C_{\mathrm{pl},3}}{2}\cdot\frac{\varepsilon_{\mathrm{pl},1}^{-1/2}}{C_{\mathrm{pl},3}\varepsilon_{\mathrm{pl},1}^{-1/2}} = \frac{1}{2}
\end{aligned}$$

for $0 \neq |\xi'| \geq R$ and $\varepsilon > 0$. In particular, Proposition 2.28 shows that

$$I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q = I + \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q - \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q$$

is continuously invertible in $L^2((-1,1);\mathbb{C}^N)$ and that the corresponding norm estimate

$$\begin{aligned}
&\big\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\qquad \le \frac{\big\|(I + \mathfrak{H}_{|\xi'|\varepsilon,\xi'/|\xi'|}Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}}{1-\frac{1}{2}} \\
&\qquad \le 2\frac{C_{\mathrm{pl},3}}{2} = C_{\mathrm{pl},3}
\end{aligned} \tag{5.38}$$

is valid for $0 \neq |\xi'| \geq R$ and $\varepsilon > 0$.

Having found an estimate for $0 \neq |\xi'| \geq R$, we aim to find a similar estimate for $0 \neq |\xi'| < R$. Again, according to Lemma 5.5 we have

$$\begin{aligned}
&\|\mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q - \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\
&\qquad \le C_{\mathrm{pl},1}\|q\|_{L^\infty((-1,1))}(|\eta|+|\tau|)\varepsilon(1+|\xi'|) = \varepsilon\frac{\varepsilon_{\mathrm{pl},1}^{-1/2}(1+|\xi'|)}{C_{\mathrm{pl},3}}
\end{aligned}$$

for $\xi' \in \mathbb{R}^{\theta-1} \setminus \{0\}$ and $\varepsilon > 0$. Moreover, Proposition 2.19, (5.29) and (5.36) imply

$$\begin{aligned}\operatorname{ess\,sup}_{\xi'\in\mathbb{R}^{\theta-1}} &\big\|(I + \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\ &= \big\|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0} \le \frac{C_{\mathrm{pl},3}}{2}.\end{aligned}$$

Hence, as $1 + |\xi'| < 1 + R = \varepsilon_{\mathrm{pl},1}^{-1/2}$, we estimate similarly as in the first part of the proof for $\varepsilon \in (0, \varepsilon_{\mathrm{pl},1})$

$$\begin{aligned}&\operatorname{ess\,sup}_{|\xi'|<R}\big\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\ &= \operatorname{ess\,sup}_{|\xi'|<R}\big\|\big[I + (I + \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big(\mathfrak{D}_{\varepsilon,\xi'}(z) - \mathfrak{D}_{0,\xi'}(z)\big)Q_{\eta,\tau}q\big]^{-1} \\ &\qquad\qquad \cdot (I + \mathfrak{D}_{0,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} \\ &< \frac{1}{1 - \frac{C_{\mathrm{pl},3}}{2}\cdot\frac{\varepsilon\varepsilon_{\mathrm{pl},1}^{-1/2}(1+R)}{C_{\mathrm{pl},3}}}\cdot\frac{C_{\mathrm{pl},3}}{2} = \frac{1}{1 - \frac{\varepsilon\varepsilon_{\mathrm{pl},1}^{-1}}{2}}\cdot\frac{C_{\mathrm{pl},3}}{2} < C_{\mathrm{pl},3}.\end{aligned} \tag{5.39}$$

Combining (5.38) and (5.39), and applying Proposition 2.19 gives us

$$\begin{aligned}&\big\|(I + D_\varepsilon^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0} \\ &\quad= \max\Big\{\operatorname{ess\,sup}_{|\xi'|\ge R}\big\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}, \\ &\qquad\qquad \operatorname{ess\,sup}_{|\xi'|<R}\big\|(I + \mathfrak{D}_{\varepsilon,\xi'}(z)Q_{\eta,\tau}q)^{-1}\big\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\Big\} \\ &\quad\le C_{\mathrm{pl},3} \qquad \forall \varepsilon \in (0, \varepsilon_{\mathrm{pl},1}).\end{aligned}$$

□

Corollary 5.9. *Let $z \in \mathbb{C} \setminus \mathbb{R}$, q be as in* (5.1)*, $S \subset \mathbb{R}^2$ be a compact set and $\max_{(\eta,\tau)\in S} \eta^2 - \tau^2 < \frac{\pi^2}{4}$. Then, there exists an $\varepsilon_{\mathrm{pl},2} = \varepsilon_{\mathrm{pl},2}(S) > 0$ such that*

$$\sup_{(\varepsilon,y_0,(\eta,\tau),\kappa)\in(0,\varepsilon_{\mathrm{pl},2})\times\mathbb{R}\times S\times\mathrm{SO}(\theta)} \big\|(I + D_\varepsilon^{y_0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0} < \infty.$$

Proof. Since $D_\varepsilon^{y_0,\kappa}(z) = D_\varepsilon^{0,\kappa}(z)$, see the text below (5.20), the assertion follows directly from Proposition 5.8 if we can show

$$\sup_{((\eta,\tau),\kappa)\in S\times\mathrm{SO}(\theta)} C_{\mathrm{pl},3}(\eta,\tau,\kappa) < \infty \quad \text{and} \quad \inf_{((\eta,\tau),\kappa)\in S\times\mathrm{SO}(\theta)} \varepsilon_{\mathrm{pl},1}(\eta,\tau,\kappa) > 0, \tag{5.40}$$

with $C_{\mathrm{pl},3}$ and $\varepsilon_{\mathrm{pl},1}$ as in Proposition 5.8. Note also that as S is bounded, the first inequality in (5.40) and (5.37) imply the second inequality in (5.40). Moreover, the assumption $\max_{(\eta,\tau)\in S} \eta^2 - \tau^2 < \frac{\pi^2}{4}$ implies $\max_{(\eta,\tau)\in S} C_{\mathrm{pl},2}(\eta,\tau) < \infty$, where $C_{\mathrm{pl},2}$ is defined by (5.35). Hence, it follows from (5.36) that (5.40) is valid if

$$\sup_{((\eta,\tau),\kappa)\in S\times\mathrm{SO}(\theta)} \big\|(I + D_0^{0,\kappa}(z)Q_{\eta,\tau}q)^{-1}\big\|_{0\to 0} < \infty. \tag{5.41}$$

By the representation of $D_0^{0,\kappa}(z)$ in Proposition 5.2 and (5.21), $D_0^{0,\kappa}(z)Q_{\eta,\tau}q$ depends with respect to the operator norm in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ continuously on η, τ and κ. Thus, as $S \times \mathrm{SO}(\theta)$ is compact, (5.41) is indeed true. □

5.1.2 General rotated C_b^2-graphs

After treating the case of affine hyperplanes, we turn to the case where Σ is a rotated C_b^2-graph. To do so, we fix in this section

$$\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R}) \quad \text{and} \quad \kappa \in \mathrm{SO}(\mathbb{R}^\theta), \tag{5.42}$$

and we assume

$$\Sigma = \Sigma_{\zeta,\kappa} = \{\kappa(x', \zeta(x')) : x' \in \mathbb{R}^{\theta-1}\}. \tag{5.43}$$

Before we resume, let us mention that in the current section we use the upper index notation introduced above (5.11), cf. (5.14) and (5.18), only for objects which do not correspond to the fixed rotated graph $\Sigma = \Sigma_{\zeta,\kappa}$; i.e. we write $\overline{B}_\varepsilon(z)$, ν, η, etc. instead of $\overline{B}_\varepsilon^{\Sigma_{\zeta,\kappa}}(z)$, $\nu^{\Sigma_{\zeta,\kappa}}$, $\eta^{\Sigma_{\zeta,\kappa}}$, etc., respectively.

Recall from (5.14) that $\varkappa_{\zeta,\kappa}(x') = \kappa(x', \zeta(x'))$ and $\nu_{\zeta,\kappa}(x') = \nu(\varkappa_{\zeta,\kappa}(x'))$ for $x' \in \mathbb{R}^{\theta-1}$. According to Proposition 2.9 (i) (in the current case we have $p = 1$, $\varkappa_1 = \varkappa_{\zeta,\kappa}$, $\Sigma_1 = \Sigma = \Sigma_{\zeta,\kappa}$), there exists a $C_{\iota,1} > 0$ such that for all $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC})$ and $x', y' \in \mathbb{R}^{\theta-1}$

$$C_{\iota,1}^{-1}\big(|x'-y'| + |\widetilde{\varepsilon}|\big) \le |\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')| \le C_{\iota,1}\big(|x'-y'| + |\widetilde{\varepsilon}|\big) \tag{5.44}$$

with $\varepsilon_{ABC} > 0$ from (4.19). Furthermore, combining the estimates from Proposition 3.4 for G_z, $z \in \rho(H)$, with (5.44) gives us for all $x', y' \in \mathbb{R}^{\theta-1}$, $j \in \{1, \dots, \theta\}$ and $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC})$ the inequalities

$$\begin{aligned} |G_z(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))| &\le C_{G,1} C_{\iota,1}^{\theta-1} (|x'-y'| + |\widetilde{\varepsilon}|)^{1-\theta} e^{-\frac{C_{G,2}}{C_{\iota,1}}|x'-y'|}, \\ |\partial_j G_z(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))| &\le C_{G,1} C_{\iota,1}^{\theta} (|x'-y'| + |\widetilde{\varepsilon}|)^{-\theta} e^{-\frac{C_{G,2}}{C_{\iota,1}}|x'-y'|}. \end{aligned} \tag{5.45}$$

We are going to prove the uniform boundedness of $(I + B_\varepsilon(z)Vq)^{-1}$ in $\mathcal{B}^0(\Sigma)$ with respect to $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$ for a suitable $\varepsilon_{\mathrm{conv}} \in (0, \varepsilon_{ABC}]$. According to (5.8), (5.10) and (5.11) this is equivalent to proving the uniform boundedness of $(I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q)^{-1}$ in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$.

We start by analysing $D_\varepsilon^{\zeta,\kappa}(z)$ locally. To proceed, we need to introduce further notations. For $x_0' \in \mathbb{R}^{\theta-1}$ we define

$$\zeta_{x_0'}(x') := \zeta(x_0') + \langle \nabla\zeta(x_0'), x' - x_0' \rangle, \qquad x' \in \mathbb{R}^{\theta-1}. \tag{5.46}$$

Moreover, we define the localization parameter $a_\varepsilon := \varepsilon^{1/6}$ for $\varepsilon \in (0, \varepsilon_{ABC})$. Next, we introduce a family of auxiliary operators. For this, we choose a C^∞-function ω with $0 \le \omega \le 1$, $\omega = 1$ on $\mathbb{R}^{\theta-1} \backslash B(0,1)$ and $\omega = 0$ on $B(0, 1/2)$. We use this function to cut out the singular part of the kernel of $D_\varepsilon^{\zeta,\kappa}(z)$; cf. (5.15). More precisely, by analogy with (5.16) and (5.17), we define for $\varepsilon \in (0, \varepsilon_{ABC})$ and $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\}$ the operators

$$\begin{aligned} e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z) &: L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N), \\ e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)g(x') &:= \int_{\mathbb{R}^{\theta-1}} G_z(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\omega\big(\tfrac{x'-y'}{a_\varepsilon}\big) \\ &\qquad\qquad \cdot \sqrt{1+|\nabla\zeta(y')|^2}\, g(y')\, dy', \end{aligned} \tag{5.47}$$

and

$$E_\varepsilon(z) : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \qquad E_\varepsilon(z)f(t) := \int_{-1}^{1} e_{\varepsilon(t-s)}^{a_\varepsilon}(z)f(s)\, ds. \tag{5.48}$$

First, we prove preliminary results for $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ and $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$. Afterwards, we transfer these results to the operators $E_\varepsilon(z)$ and $D_\varepsilon^{\zeta,\kappa}(z)$ in Proposition 5.14.

Lemma 5.10. *Let $z \in \rho(H)$, $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\}$, $\varepsilon \in (0, \varepsilon_{ABC})$ as well as $a_\varepsilon = \varepsilon^{1/6}$. Then, the operator $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ acts as a bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and*

$$\|e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon},$$

where $C > 0$ does not depend on $\widetilde{\varepsilon}$ and ε. Moreover, for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ the mapping $(-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\} \ni \widetilde{\varepsilon} \mapsto e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)f \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ is continuous.

Proof. We aim to prove the assertion by applying Lemma C.1. To do so, it is necessary to find suitable estimates for the kernel of $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ which is for $x', y' \in \mathbb{R}^{\theta-1}$ given by

$$k(x',y') := G_z(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\omega\big(\tfrac{x'-y'}{a_\varepsilon}\big)\sqrt{1+|\nabla\zeta(y')|^2}.$$

We notice as $G_z \in C^\infty(\mathbb{R}^\theta \setminus \{0\};\mathbb{C}^{N\times N})$, $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and $\omega \in C_b^\infty(\mathbb{R}^{\theta-1};\mathbb{R})$, and as ω cuts out the singularity of G_z, we have $k \in C_b^1(\mathbb{R}^{\theta-1}\times\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$. Furthermore, using (5.45), $0 \leq \omega \leq 1$, $\operatorname{supp}\omega \subset \mathbb{R}^{\theta-1} \setminus B(0,1/2)$ and $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ immediately gives us for $x' \neq y' \in \mathbb{R}^{\theta-1}$

$$\begin{aligned} |k(x',y')| &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\tfrac{x'-y'}{a_\varepsilon}\big)(|x'-y'| + |\widetilde{\varepsilon}|)^{1-\theta}e^{-c|x'-y'|} \\ &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}\big(\tfrac{x'-y'}{a_\varepsilon}\big)|x'-y'|^{1-\theta}e^{-c|x'-y'|}, \end{aligned}$$

where $c = \frac{C_{G,2}}{C_{\iota,1}}$ with $C_{\iota,1} > 0$ from (5.44) and $C_{G,2}$ from Proposition 3.4. Next, we estimate the derivatives of k. The l-th derivative, $l \in \{1, \ldots, \theta-1\}$, with respect to

x' is given by

$$\begin{aligned}\frac{d}{dx'_l}k(x',y') = \sum_{j=1}^{\theta}\Big(&(\partial_j G_z)(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\\ &\cdot \frac{d}{dx'_l}(\varkappa_{\zeta,\kappa}(x')[j] + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')[j])\omega(\tfrac{x'-y'}{a_\varepsilon})\sqrt{1+|\nabla\zeta(y')|^2}\Big)\\ &+ G_z(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\frac{1}{a_\varepsilon}(\partial_l\omega)(\tfrac{x'-y'}{a_\varepsilon})\sqrt{1+|\nabla\zeta(y')|^2}\end{aligned}$$

for $x' \neq y' \in \mathbb{R}^{\theta-1}$, where $v[j]$ denotes the j-th component of a vector v. Applying (5.45), and the properties of ζ and ω again lets us estimate

$$\begin{aligned}\Big|\frac{d}{dx'_l}k(x',y')\Big| &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}(\tfrac{x'-y'}{a_\varepsilon})\Big((|x'-y'|+|\widetilde{\varepsilon}|)^{-\theta}e^{-c|x'-y'|}\\ &\qquad + \frac{1}{a_\varepsilon}(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c|x'-y'|}\Big)\\ &\leq C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}(\tfrac{x'-y'}{a_\varepsilon})\Big(|x'-y'|^{-\theta} + \frac{1}{a_\varepsilon}|x'-y'|^{1-\theta}\Big)e^{-c|x'-y'|}.\end{aligned}$$

Thus, if we set $\widetilde{k}(z') := C\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}(\frac{z'}{a_\varepsilon})\Big(|z'|^{-\theta}+\frac{1}{a_\varepsilon}|z'|^{1-\theta}\Big)e^{-c|z'|}$ for $z' \in \mathbb{R}^{\theta-1}\setminus\{0\}$ we get

$$|k(x',y')|, \sum_{l=1}^{\theta-1}\Big|\frac{d}{dx'_l}k(x',y')\Big| \leq \widetilde{k}(x'-y') \quad \forall x' \neq y' \in \mathbb{R}^{\theta-1}. \tag{5.49}$$

Hence, $e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)$ acts by Lemma C.1 as a bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and

$$\|e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}.$$

Now, the norm estimate in the assertion follows from

$$\begin{aligned}\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})} &= C\int_{\mathbb{R}^{\theta-1}}\chi_{\mathbb{R}^{\theta-1}\setminus B(0,1/2)}(\tfrac{z'}{a_\varepsilon})\Big(|z'|^{-\theta}+\frac{1}{a_\varepsilon}|z'|^{1-\theta}\Big)e^{-c|z'|}dz'\\ &\leq C\int_{a_\varepsilon/2}^{\infty}\Big(r^{-\theta}+\frac{1}{a_\varepsilon}r^{1-\theta}\Big)e^{-cr}r^{\theta-2}\,dr\\ &\leq C\Big(\frac{1}{a_\varepsilon}+\frac{1+|\log(a_\varepsilon)|}{a_\varepsilon}\Big)\\ &\leq C\frac{1+|\log(a_\varepsilon)|}{a_\varepsilon} \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon}.\end{aligned}$$

Finally, we prove the continuity. For this, let $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC})\setminus\{0\}$ and $(\widetilde{\varepsilon}_n)_{n\in\mathbb{N}}$ be a sequence such that $\widetilde{\varepsilon}_n \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC})\setminus\{0\}$ for all $n \in \mathbb{N}$ and $\widetilde{\varepsilon}_n \to \widetilde{\varepsilon}$

as $n \to \infty$. Using the dominated convergence theorem and (5.49) shows that for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ $e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}(z)f$ and $\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}(z)f$, $l \in \{1,\dots,\theta-1\}$, converge pointwise to $e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)f$ and $\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)f$, $l \in \{1,\dots,\theta-1\}$, respectively. Furthermore, the estimate from (5.49) shows that the functions $|e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}(z)f|$ and $|\partial_l e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}(z)f|$, $l \in \{1,\dots,\theta-1\}$, are independently of $n \in \mathbb{N}$ pointwise bounded by the function $|f| * \widetilde{k}$ which is by Young's inequality square integrable as $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and $\widetilde{k} \in L^1(\mathbb{R}^{\theta-1})$. Hence, applying the dominated converge theorem again shows that $e^{a_\varepsilon}_{\widetilde{\varepsilon}_n}(z)f$ converges to $e^{a_\varepsilon}_{\widetilde{\varepsilon}}(z)f$ in $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. □

Lemma 5.11. *Let $x_0' \in \mathbb{R}^{\theta-1}$, $\zeta_{x_0'}$ be as in* (5.46)*, $z \in \rho(H)$, $\psi \in C_b^1(\mathbb{R}^{\theta-1})$ and $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\}$. Then,*

$$\big\|[d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log|\widetilde{\varepsilon}||),$$

where $C > 0$ does not depend on $\widetilde{\varepsilon}$ and $x_0' \in \mathbb{R}^{\theta-1}$. Moreover, for $f \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ the mapping $(-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\} \ni \widetilde{\varepsilon} \mapsto [d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]f \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ is continuous.

Proof. We prove this result in the same vein as the previous lemma, i.e. we estimate the kernel of $[d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]$ and its partial derivatives, and apply Lemma C.1. The kernel of $[d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]$ is given by

$$k(x',y') := G_z(\varkappa_{\zeta_{x_0'},\kappa}(x') - \varkappa_{\zeta_{x_0'},\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta_{x_0'},\kappa}(x'))\sqrt{1+|\nabla\zeta_{x_0'}(y')|^2}(\psi(y')-\psi(x'))$$

for $x', y' \in \mathbb{R}^{\theta-1}$. The representations

$$\begin{aligned} \varkappa_{\zeta_{x_0'},\kappa}(x') &= \kappa(x', \zeta(x_0') + \langle\nabla\zeta(x_0'), x'-x_0'\rangle), \\ \nu_{\zeta_{x_0'},\kappa}(x') &= \frac{\kappa(-\nabla\zeta(x_0'),1)}{\sqrt{1+|\nabla\zeta(x_0')|^2}} = \nu_{\zeta,\kappa}(x_0'), \\ \nabla\zeta_{x_0'}(x') &= \nabla\zeta(x_0'), \end{aligned} \tag{5.50}$$

for $x' \in \mathbb{R}^{\theta-1}$ show that k can be simplified to

$$k(x',y') := G_z(\kappa(x'-y', \langle\nabla\zeta(x_0'), x'-y'\rangle) + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))\sqrt{1+|\nabla\zeta(x_0')|^2}(\psi(y')-\psi(x')).$$

Moreover, with (5.50) and $\kappa \in \mathrm{SO}(\theta)$ one gets

$$\begin{aligned} |\kappa(x'-y', \langle\nabla\zeta(x_0'), x'-y'\rangle) &+ \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')|^2 \\ &= |x'-y'|^2 + \langle\nabla\zeta(x_0'), x'-y'\rangle^2 + \widetilde{\varepsilon}^2 \\ &\le |x'-y'|^2(1+\|\nabla\zeta\|^2_{L^\infty(\mathbb{R}^{\theta-1};\mathbb{R}^{\theta-1})}) + \widetilde{\varepsilon}^2. \end{aligned}$$

In particular, we can choose a $C_{\text{lin}} > 0$ which does not depend on x'_0 and $\widetilde{\varepsilon}$ such that

$$\begin{aligned}(C_{\text{lin}})^{-1}(|x'-y'|+|\widetilde{\varepsilon}|) &\le |\kappa(x'-y',\langle\nabla\zeta(x'_0),x'-y'\rangle)+\widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'_0)| \\ &\le C_{\text{lin}}(|x'-y'|+|\widetilde{\varepsilon}|).\end{aligned} \tag{5.51}$$

Then, Proposition 3.4, (5.51), $\psi \in C^1_b(\mathbb{R}^{\theta-1})$ and $\zeta \in C^2_b(\mathbb{R}^{\theta-1};\mathbb{R})$ yield

$$\begin{aligned}|k(x',y')| &\le C(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|}\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}|x'-y'| \\ &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(|x'-y'|+|\widetilde{\varepsilon}|)^{2-\theta}e^{-c'|x'-y'|} \quad \forall x',y'\in\mathbb{R}^{\theta-1},\end{aligned}$$

where $c' = \frac{C_{G,2}}{C_{\text{lin}}}$ with $C_{G,2}$ from Proposition 3.4. The l-th derivative, $l \in \{1,\dots,\theta-1\}$, with respect to $x' \in \mathbb{R}^{\theta-1}$ of k is given by

$$\begin{aligned}\frac{d}{dx'_l}k(x',y') = \Big(&\sum_{j=1}^{\theta}(\partial_j G_z)(\kappa(x'-y',\langle\nabla\zeta(x'_0),x'-y'\rangle)+\widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'_0)) \\ &\qquad\cdot\big(\kappa(e'_l,\partial_l\zeta(x'_0))\big)[j](\psi(y')-\psi(x')) \\ &- G_z(\kappa(x'-y',\langle\nabla\zeta(x'_0),x'-y'\rangle)+\widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'_0))(\partial_l\psi)(x')\Big) \\ &\qquad\cdot\sqrt{1+|\nabla\zeta(x'_0)|^2},\end{aligned}$$

where e'_l denotes the l-th Euclidean unit vector in $\mathbb{R}^{\theta-1}$ and $\big(\kappa(e'_l,\partial_l\zeta(x'_0))\big)[j]$ denotes the j-th entry of the vector $\kappa(e'_l,\partial_l\zeta(x'_0))$. Using Proposition 3.4, (5.51), $\psi \in C^1_b(\mathbb{R}^{\theta-1})$ and $\zeta \in C^2_b(\mathbb{R}^{\theta-1};\mathbb{R})$ again gives us

$$\begin{aligned}\Big|\frac{d}{dx'_l}k(x',y')\Big| &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\Big((|x'-y'|+|\widetilde{\varepsilon}|)^{-\theta}e^{-c'|x'-y'|}|x'-y'| \\ &\qquad + (|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|}\Big) \\ &\le C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(|x'-y'|+|\widetilde{\varepsilon}|)^{1-\theta}e^{-c'|x'-y'|}\end{aligned}$$

for all $x',y' \in \mathbb{R}^{\theta-1}$, where $C > 0$ can be chosen independently of x'_0 and $\widetilde{\varepsilon}$. Setting

$$\widetilde{k}(z') = C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\big((|z'|+|\widetilde{\varepsilon}|)^{2-\theta}+(|z'|+|\widetilde{\varepsilon}|)^{1-\theta}\big)e^{-c'|z'|}$$

for $z' \in \mathbb{R}^{\theta-1}$ leads to

$$|k(x',y')|, \sum_{l=1}^{\theta-1}\Big|\frac{d}{dx'_l}k(x',y')\Big| \le \widetilde{k}(x'-y') \quad \forall x',y'\in\mathbb{R}^{\theta-1}.$$

Now, Lemma C.1 shows

$$\begin{aligned}
\big\|[d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z),\psi]\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &\leq C\int_{\mathbb{R}^{\theta-1}} \widetilde{k}(z')\,dz' \\
&\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_0^\infty \big((r+|\widetilde{\varepsilon}|)^{2-\theta}+(r+|\widetilde{\varepsilon}|)^{1-\theta}\big)e^{-c'r}r^{\theta-2}\,dr \\
&\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_0^\infty \big(1+(r+|\widetilde{\varepsilon}|)^{-1}\big)e^{-c'r}\,dr \\
&\leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log|\widetilde{\varepsilon}||).
\end{aligned}$$

The assertion regarding the continuity can be proven in a similar way as in the previous lemma. □

Lemma 5.12. *Let $x_0'\in\mathbb{R}^{\theta-1}$, ζ and κ be as in (5.42), $\zeta_{x_0'}$ be as in (5.46), $z\in\rho(H)$, $a_\varepsilon=\varepsilon^{1/6}$ and $\widetilde{\varepsilon}\in(-2\varepsilon_{ABC},2\varepsilon_{ABC})\setminus\{0\}$. Then, there exists an $\varepsilon_{\mathrm{gr},1}=\varepsilon_{\mathrm{gr},1}(\zeta)$ in the interval $(0,\varepsilon_{ABC}]$ such that for all $\varepsilon\in(0,\varepsilon_{\mathrm{gr},1})$ the inequality*

$$\big\|\chi_{B(x_0',3a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)-d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z)\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\leq Ca_\varepsilon(1+|\log|\widetilde{\varepsilon}||)$$

holds, where $C>0$ does not depend on ε, $\widetilde{\varepsilon}$ and x_0'.

Proof. We prove this statement by estimating the kernel of the operator

$$\chi_{B(x_0',a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)-d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z)\big)\chi_{B(x_0',a_\varepsilon)}$$

and applying Lemma C.1. The mentioned kernel is given by

$$\begin{aligned}
k(x',y') := \chi_{B(x_0',3a_\varepsilon)}(x')\Bigg(&G_z(\varkappa_{\zeta,\kappa}(x')-\varkappa_{\zeta,\kappa}(y')+\widetilde{\varepsilon}\nu_{\zeta,\kappa}(x'))\sqrt{1+|\nabla\zeta(y')|^2} \\
&-G_z(\kappa(x'-y',\langle\nabla\zeta(x_0'),x'-y'\rangle)+\widetilde{\varepsilon}\nu_{\zeta,\kappa}(x_0'))\sqrt{1+|\nabla\zeta(x_0')|^2}\Bigg) \\
&\cdot\chi_{B(x_0',3a_\varepsilon)}(y')
\end{aligned}$$

for $x'\neq y'\in\mathbb{R}^{\theta-1}$. If $x'\notin B(x_0',3a_\varepsilon)$ or $y'\notin B(x_0',3a_\varepsilon)$, then $k(x',y')=0$. Thus, we assume from now on $x',y'\in B(x_0',3a_\varepsilon)$. Using Lemma 2.8, Proposition 3.4 and $\zeta\in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ gives us

$$\begin{aligned}
|k(x',y')| &\leq \sqrt{\theta}\sup_{v\in[0,1],j\in\{1,\dots\theta\}}\big|\partial_j G_z(w_v)\sqrt{1+|\nabla\zeta(y')|^2}\big||w_1-w_0| \\
&\qquad + \big|G_z(w_0)\big(\sqrt{1+|\nabla\zeta(y')|^2}-\sqrt{1+|\nabla\zeta(x_0')|^2}\big)\big| \qquad (5.52)\\
&\leq C\Big(\sup_{v\in[0,1]}|w_v|^{-\theta}|w_1-w_0|+|w_0|^{1-\theta}|x_0'-y'|\Big)
\end{aligned}$$

with

$$w_\upsilon = \upsilon\big(\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x')\big) + (1-\upsilon)\big(\kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')\big)$$

for $\upsilon \in [0,1]$. We remark that we were able to apply Lemma 2.8 since $w_\upsilon \neq 0$ for all $\upsilon \in [0,1]$, which turns out be true in the course of the proof; cf. (5.54). Next, let us estimate

$$\begin{aligned}|w_1 - w_0| &= |\varkappa_{\zeta,\kappa}(x') - \varkappa_{\zeta,\kappa}(y') + \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x') - \kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) - \widetilde{\varepsilon}\nu_{\zeta,\kappa}(x_0')| \\ &= |\kappa(x'-y', \zeta(x') - \zeta(y')) \\ &\qquad - \kappa(x'-y', \langle \nabla\zeta(x_0'), x'-y'\rangle) + \widetilde{\varepsilon}(\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0'))| \\ &\le |\kappa(0, \zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle)| + |\widetilde{\varepsilon}(\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0'))|.\end{aligned}$$

As $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ and $\kappa \in \mathrm{SO}(\theta)$, there exists a $C_\zeta > 0$ such that

$$|\nu_{\zeta,\kappa}(x') - \nu_{\zeta,\kappa}(x_0')| = \left| \frac{\kappa(-\nabla\zeta(x'),1)}{\sqrt{1+|\nabla\zeta(x')|^2}} - \frac{\kappa(-\nabla\zeta(x_0'),1)}{\sqrt{1+|\nabla\zeta(x_0')|^2}} \right| \le C_\zeta|x'-x_0'| \le 3C_\zeta a_\varepsilon$$

and

$$\begin{aligned}&|\kappa(0, \zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle)| \\ &\qquad = |\zeta(x') - \zeta(y') - \langle \nabla\zeta(x_0'), x'-y'\rangle| \\ &\qquad = \Big| \int_0^1 \langle \nabla\zeta(y' + t(x'-y')) - \nabla\zeta(x_0'), x'-y'\rangle \, dt \Big| \\ &\qquad \le C_\zeta \Big| \int_0^1 |t(x'-x_0') + (1-t)(y'-x_0')||x'-y'| \, dt \Big| \\ &\qquad \le 3C_\zeta a_\varepsilon |x'-y'|,\end{aligned}$$

where we used $x', y' \in B(x_0', 3a_\varepsilon)$. Hence, if $\varepsilon_{\mathrm{gr},1} = \varepsilon_{\mathrm{gr},1}(\zeta) > 0$ is chosen sufficiently small, then for all $a_\varepsilon \in (0, \varepsilon_{\mathrm{gr},1}^{1/6})$ the inequality

$$|w_1 - w_0| \le 3C_\zeta a_\varepsilon(|x'-y'| + |\widetilde{\varepsilon}|) \le \frac{1}{2C_{\iota,1}}(|x'-y'| + |\widetilde{\varepsilon}|) \tag{5.53}$$

holds with $C_{\iota,1} > 0$ from (5.44). Therefore, we can use (5.44) to estimate $|w_\upsilon|$, $\upsilon \in [0,1]$, from below by

$$\begin{aligned}|w_\upsilon| &= |\upsilon w_1 + (1-\upsilon)w_0| = |w_1 + (1-\upsilon)(w_0 - w_1)| \ge |w_1| - |w_1 - w_0| \\ &\ge \frac{1}{C_{\iota,1}}(|x'-y'| + |\widetilde{\varepsilon}|) - \frac{1}{2C_{\iota,1}}(|x'-y'| + |\widetilde{\varepsilon}|) = \frac{1}{2C_{\iota,1}}(|x'-y'| + |\widetilde{\varepsilon}|).\end{aligned} \tag{5.54}$$

Thus, by plugging (5.53) and (5.54) into (5.52), the kernel k can be estimated for $a_\varepsilon = \varepsilon^{1/6}$ with $\varepsilon \in (0, \varepsilon_{\mathrm{gr},1})$ by

$$|k(x',y')| \le \begin{cases} 0, & x' \notin B(x_0', 3a_\varepsilon) \text{ or } y' \notin B(x_0', 3a_\varepsilon), \\ Ca_\varepsilon(|x'-y'| + |\widetilde{\varepsilon}|)^{1-\theta}, & x' \in B(x_0', 3a_\varepsilon) \text{ and } y' \in B(x_0', 3a_\varepsilon). \end{cases}$$

Applying Lemma C.1 yields

$$\begin{aligned}
\big\|\chi_{B(x_0',3a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z)\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}&\\
\leq C a_\varepsilon \int_{B(0,6a_\varepsilon)} (|z'| + |\widetilde{\varepsilon}|)^{1-\theta}\, dz'&\\
\leq C a_\varepsilon \int_0^{6a_\varepsilon} (r + |\widetilde{\varepsilon}|)^{1-\theta} r^{\theta-2}\, dr&\\
\leq C a_\varepsilon \int_0^{6a_\varepsilon} (r + |\widetilde{\varepsilon}|)^{-1}\, dr&\\
\leq C a_\varepsilon (1 + |\log|\widetilde{\varepsilon}||).&
\end{aligned}$$

□

Corollary 5.13. *Let $x_0' \in \mathbb{R}^{\theta-1}$, ζ and κ be as in (5.42), $\zeta_{x_0'}$ be as in (5.46), Σ be as in (5.43), $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (5.18), $z \in \rho(H)$, $\varepsilon \in (0, \varepsilon_{\mathrm{gr},1})$ with $\varepsilon_{\mathrm{gr},1}$ chosen as in the previous lemma, $a_\varepsilon = \varepsilon^{1/6}$ and $\widetilde{\varepsilon} \in (-2\varepsilon_{ABC}, 2\varepsilon_{ABC}) \setminus \{0\}$. Then,*

$$\begin{aligned}
&\big\|\chi_{B(x_0',3a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}\\
&\quad - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C a_\varepsilon(1 + |\log|\widetilde{\varepsilon}||),
\end{aligned}$$

where $C > 0$ does not depend on ε, $\widetilde{\varepsilon}$ and x_0'.

Proof. The previous lemma and $Q_{\eta,\tau}^{\zeta,\kappa} \in C_b^1(\mathbb{R}^{\theta-1}; \mathbb{C}^{N\times N})$ yield

$$\begin{aligned}
&\big\|\chi_{B(x_0',3a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\\
&\leq \big\|\chi_{B(x_0',3a_\varepsilon)} d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)\big(Q_{\eta,\tau}^{\zeta,\kappa} - Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\\
&\quad + \big\|\chi_{B(x_0',3a_\varepsilon)}\big(d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z) - d_{\widetilde{\varepsilon}}^{\zeta_{x_0'},\kappa}(z)\big)\chi_{B(x_0',3a_\varepsilon)} Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\\
&\leq C\big(\|d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} a_\varepsilon + a_\varepsilon(1 + |\log|\widetilde{\varepsilon}||)\big),
\end{aligned}$$

where $C > 0$ does not depend on ε, $\widetilde{\varepsilon}$ and x_0'. Moreover, (5.17), (5.44), (5.45) and $\zeta \in C_b^2(\mathbb{R}^{\theta-1}; \mathbb{R})$ let us estimate the kernel k of $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$ by

$$|k(x', y')| \leq C(|x' - y'| + |\widetilde{\varepsilon}|)^{1-\theta} e^{-c|x'-y'|} \quad \forall x', y' \in \mathbb{R}^{\theta-1},$$

where $c = \frac{C_{G,2}}{C_{\iota,1}} > 0$ with $C_{\iota,1}$ from (5.44) and $C_{G,2}$ from Proposition 3.4, and therefore Lemma C.1 implies

$$\begin{aligned}
\|d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} &\leq C \int_{\mathbb{R}^{\theta-1}} (|z'| + |\widetilde{\varepsilon}|)^{1-\theta} e^{-c|z'|}\, dz'\\
&\leq C \int_0^\infty (r + |\widetilde{\varepsilon}|)^{1-\theta} e^{-cr} r^{\theta-2}\, dr\\
&\leq C(1 + |\log|\widetilde{\varepsilon}||),
\end{aligned}$$

which yields the assertion. □

Now, we can transfer the results for $d_{\widetilde{\varepsilon}}^{\zeta,\kappa}(z)$ and $e_{\widetilde{\varepsilon}}^{a_\varepsilon}(z)$ from Lemma 5.10, Lemma 5.11 and Corollary 5.13 to the operators $D_{\varepsilon}^{\zeta,\kappa}(z)$ and $E_\varepsilon(z)$ via their representations in (5.16) and (5.48), respectively. We obtain the following statement.

Proposition 5.14. *Let $x_0' \in \mathbb{R}^{\theta-1}$, ζ and κ be as in (5.42), Σ be as in (5.43), $\zeta_{x_0'}$ be as in (5.46), $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (5.18), $z \in \rho(H)$, $\psi \in C_b^1(\mathbb{R}^{\theta-1})$, $\varepsilon \in (0, \varepsilon_{\mathrm{gr},1})$ with $\varepsilon_{\mathrm{gr},1}$ chosen as in Lemma 5.12 and $a_\varepsilon = \varepsilon^{1/6}$. Then, the operators $E_\varepsilon(z)$ and $[D_\varepsilon^{\zeta,\kappa}(z), \psi]$ act as bounded operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$ and*

$$\|E_\varepsilon(z)\|_{0\to 1} \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon},$$

$$\big\|[D_\varepsilon^{\zeta_{x_0'},\kappa}(z), \psi]\big\|_{0\to 1} \leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|),$$

$$\big\|\chi_{B(x_0',3a_\varepsilon)}\big(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{0\to 0} \leq Ca_\varepsilon(1+|\log(\varepsilon)|),$$

where $C > 0$ does not depend on x_0' and ε.

Proof. First, we consider $[D_\varepsilon^{\zeta_{x_0'},\kappa}(z), \psi]$. We start by showing that $[D_\varepsilon^{\zeta_{x_0'},\kappa}(z), \psi]$ is well-defined as an operator from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Let $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$ and set $g = [D_\varepsilon^{\zeta_{x_0'},\kappa}(z), \psi]f$. It follows from (5.16) that g has the representation

$$g(t) = \int_{-1}^{1} [d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z), \psi]f(s)\,ds, \qquad t \in (-1,1).$$

Since $[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z), \psi]$ has the continuity property from Lemma 5.11 and $f \in \mathcal{B}^0(\mathbb{R}^{\theta-1})$, the text below Definition 2.13 implies that the function

$$(-1,1)\times(-1,1) \ni (t,s) \mapsto [d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z), \psi]f(s) \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$$

is measurable. According to Lemma 5.11 we have

$$\begin{aligned}
&\int_{-1}^{1}\Big(\int_{-1}^{1} \|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z), \psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\Big)^2 dt \\
&\leq C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_{-1}^{1}\Big(\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\|f(s)\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\Big)^2 dt.
\end{aligned} \tag{5.55}$$

This expression can be estimated with the Cauchy-Schwarz inequality and Fubini's theorem by

$$\begin{aligned}
&C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\int_{-1}^{1}\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\,ds \\
&\qquad\qquad \cdot\int_{-1}^{1}(1+|\log|\varepsilon(t-s)||)\|f(s)\|^2_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\,dt \\
&\leq C\|\psi\|^2_{W^1_\infty(\mathbb{R}^{\theta-1})}\Big(\int_{-2}^{2}(1+|\log|\varepsilon s||)\,ds\Big)^2\int_{-1}^{1}\|f(s)\|^2_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds \\
&\leq C\big(\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|)\|f\|_0\big)^2.
\end{aligned} \tag{5.56}$$

In particular, applying the Cauchy-Schwarz inequality again gives us

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} \|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\,dt \\
&\qquad \le \sqrt{2}\sqrt{\int_{-1}^{1}\Big(\int_{-1}^{1} \|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\Big)^2\,dt} < \infty.
\end{aligned}$$

Thus, (2.8) and Fubini's theorem, see Proposition 2.15, shows that $g = [D_\varepsilon^{\zeta_{x_0'},\kappa}(z),\psi]f$ is well-defined and measurable as a function from $(-1,1)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. Moreover, (5.55) and (5.56) also give us the norm estimate as

$$\begin{aligned}
\|g\|_1^2 &= \int_{-1}^{1} \|g(t)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}^2\,dt \\
&\le \int_{-1}^{1}\Big(\int_{-1}^{1} \|[d_{\varepsilon(t-s)}^{\zeta_{x_0'},\kappa}(z),\psi]f(s)\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)}\,ds\Big)^2\,dt \\
&\le C\big(\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|)\|f\|_0\big)^2.
\end{aligned}$$

The proof for $E_\varepsilon(z)$ can be done in exactly the same way. Moreover,

$$\chi_{B(x_0',3a_\varepsilon)}\big(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}$$

is a well-defined operator in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. So in this case it is sufficient to show the norm estimate, which can be proven in the same way as the norm estimate for $[D_\varepsilon^{\zeta_{x_0'},\kappa}(z),\psi]$. $\square$

As the last part of our local analysis we state a result concerning the inverse of $I + D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')q$ for $x_0' \in \mathbb{R}^{\theta-1}$. This is an important result since these operators play an essential role when constructing the inverse of $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$.

Proposition 5.15. *Let ζ and κ be as in* (5.42), *Σ be as in* (5.43), *$\zeta_{x_0'}$ be as in* (5.46), *$\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$, $d = \eta^2 - \tau^2$ satisfy*

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4},$$

q be as in (5.1), *$Q_{\eta,\tau}^{\zeta,\kappa}$ be as in* (5.18) *and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, there exists an $\varepsilon_{\mathrm{gr},2} > 0$ such that the operators $(I + D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')q)^{-1}$ are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0,\varepsilon_{\mathrm{gr},2})$ and $x_0' \in \mathbb{R}^{\theta-1}$.*

Proof. Note that

$$Q^{\zeta,\kappa}_{\eta,\tau}(x_0') = \eta(\varkappa_{\zeta,\kappa}(x_0'))I_N + \tau(\varkappa_{\zeta,\kappa}(x_0'))\beta = Q_{\eta(\varkappa_{\zeta,\kappa}(x_0')),\tau(\varkappa_{\zeta,\kappa}(x_0'))}$$

for $x_0' \in \mathbb{R}^{\theta-1}$; cf. (5.18) and (5.19) and the text below (5.43). Moreover, for every $x_0' \in \mathbb{R}^{\theta-1}$ the set $\Sigma_{\zeta_{x_0'},\kappa}$ is an affine hyperplane in $\mathbb{R}^\theta$ and therefore there exists a $\widetilde{y_0}(x_0') \in \mathbb{R}$ and a $\widetilde{\kappa}(x_0') \in \mathrm{SO}(\theta)$ such that

$$\begin{aligned}\Sigma_{\zeta_{x_0'},\kappa} &= \{\kappa(x', \zeta_{x_0'}(x')) : x' \in \mathbb{R}^{\theta-1}\}\\ &= \widetilde{\kappa}(x_0')\big(\mathbb{R}^{\theta-1} \times \{\widetilde{y_0}(x_0')\}\big) = \Sigma_{\widetilde{y_0}(x_0'),\widetilde{\kappa}(x_0')}.\end{aligned} \tag{5.57}$$

Hence, we get from (5.9), (5.10) and (5.11)

$$\begin{aligned}&\big\|(I + D_\varepsilon^{\zeta_{x_0'},\kappa}(z)Q^{\zeta,\kappa}_{\eta,\tau}(x_0')q)^{-1}\big\|_{0\to 0}\\ &\quad= \big\|(I + \overline{B}_\varepsilon^{\Sigma_{\zeta_{x_0'},\kappa}}(z)Q_{\eta(\varkappa_{\zeta,\kappa}(x_0')),\tau(\varkappa_{\zeta,\kappa}(x_0'))}q)^{-1}\big\|_{0\to 0}\\ &\quad= \big\|(I + \overline{B}_\varepsilon^{\Sigma_{\widetilde{y_0}(x_0'),\widetilde{\kappa}(x_0')}}(z)Q_{\eta(\varkappa_{\zeta,\kappa}(x_0')),\tau(\varkappa_{\zeta,\kappa}(x_0'))}q)^{-1}\big\|_{0\to 0}\\ &\quad= \big\|(I + D_\varepsilon^{\widetilde{y_0}(x_0'),\widetilde{\kappa}(x_0')}(z)Q_{\eta(\varkappa_{\zeta,\kappa}(x_0')),\tau(\varkappa_{\zeta,\kappa}(x_0'))}q)^{-1}\big\|_{0\to 0}.\end{aligned}$$

Now, the result follows from applying Corollary 5.9 (for $S = \overline{\mathrm{ran}\,(\eta,\tau)}$) if one chooses $\varepsilon_{\mathrm{gr},2} = \varepsilon_{\mathrm{pl},2}(\overline{\mathrm{ran}\,(\eta,\tau)}) > 0$, where $\varepsilon_{\mathrm{pl},2}$ was introduced in Corollary 5.9. □

Inspired by the local principle in [60, Proposition 5], see also [59, 61], we construct partitions of unity which allow us to globalize the established local results. We start by choosing a partition of unity $(\phi_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ for $\mathbb{R}^{\theta-1}$ with uniformly bounded derivatives which satisfies $\operatorname{supp}\phi_{n'} \subset B(n',1)$ for $n' \in \mathbb{Z}^{\theta-1}$. Moreover, let $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a sequence of functions with uniformly bounded derivatives which fulfils $0 \leq \vartheta_{n'} \leq 1$, $\vartheta_{n'} = 1$ on $B(n',2)$ and $\operatorname{supp}\vartheta_{n'} \subset B(n',3)$ for $n' \in \mathbb{Z}^{\theta-1}$. According to Proposition A.2 such sequences exist. By defining for $a \in (0,(\varepsilon_{ABC})^{1/6})$ and $n' \in \mathbb{Z}^{\theta-1}$ the functions $\phi^a_{n'}(\cdot) = \phi_{n'}(\cdot/a)$ and $\vartheta^a_{n'}(\cdot) = \vartheta_{n'}(\cdot/a)$ we obtain similar sequences; in particular $(\phi^a_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is a partition of unity for $\mathbb{R}^{\theta-1}$ with scaled supports. Furthermore, there exists a $C > 0$ which does not depend on a such that

$$\sup_{n'\in\mathbb{Z}^{\theta-1}} \max\{\|\phi^a_{n'}\|_{W^1_\infty(\mathbb{R}^{\theta-1})}, \|\vartheta^a_{n'}\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\} < \frac{C}{a}; \tag{5.58}$$

cf. Corollary A.3. Before we use these essential observations in the proof of Proposition 5.17 to construct the right inverse of $I + D^{\zeta,\kappa}_\varepsilon(z)Q^{\zeta,\kappa}_{\eta,\tau}q$, we state a helpful preliminary lemma.

Lemma 5.16. *Let $z \in \rho(H)$, $\varepsilon \in (0,\varepsilon_{ABC})$ and $a_\varepsilon = \varepsilon^{1/6}$. Then,*

$$(1 - \vartheta^{a_\varepsilon}_{n'})E_\varepsilon(z)\phi^{a_\varepsilon}_{n'} = (1 - \vartheta^{a_\varepsilon}_{n'})D^{\zeta,\kappa}_\varepsilon(z)\phi^{a_\varepsilon}_{n'}.$$

Proof. We prove this lemma by showing that the difference of the integral kernels of $(1-\vartheta_{n'}^{a_\varepsilon})E_\varepsilon(z)\phi_{n'}^{a_\varepsilon}$ and $(1-\vartheta_{n'}^{a_\varepsilon})D_\varepsilon^{\zeta,\kappa}(z)\phi_{n'}^{a_\varepsilon}$ is zero. According to (5.15), (5.47) and (5.48) this difference is given by

$$\begin{aligned}&(1-\vartheta_{n'}^{a_\varepsilon}(x'))\big(\omega(\tfrac{x'-y'}{a_\varepsilon})-1\big)\phi_{n'}^{a_\varepsilon}(y')\\&\qquad\cdot G_z(\varkappa_{\zeta,\kappa}(x')-\varkappa_{\zeta,\kappa}(y')+\varepsilon(t-s)\nu_{\zeta,\kappa}(x'))\sqrt{1+|\nabla\zeta(y')|^2}\end{aligned}\tag{5.59}$$

for all $x', y' \in \mathbb{R}^{\theta-1}$ and $t, s \in (-1,1)$. If $y' \notin B(a_\varepsilon n', a_\varepsilon)$ for an $n' \in \mathbb{R}^{\theta-1}$, then $\frac{y'}{a_\varepsilon} \notin B(n',1) \supset \operatorname{supp}\phi_{n'}$ and therefore

$$\phi_{n'}^{a_\varepsilon}(y') = \phi_{n'}(\tfrac{y'}{a_\varepsilon}) = 0.$$

Furthermore, if $x' \in B(a_\varepsilon n', 2a_\varepsilon)$, then $\frac{x'}{a_\varepsilon} \in B(n',2)$ and hence as $\vartheta_{n'} = 1$ on $B(n',2)$ we have

$$1-\vartheta_{n'}^{a_\varepsilon}(x') = 1-\vartheta_{n'}(\tfrac{x'}{a_\varepsilon}) = 0.$$

These two observations show that if $y' \notin B(a_\varepsilon n', a_\varepsilon)$ or $x' \in B(a_\varepsilon n', 2a_\varepsilon)$, then (5.59) vanishes. Thus, it remains to consider the case $y' \in B(a_\varepsilon n', a_\varepsilon)$ and $x' \notin B(a_\varepsilon n', 2a_\varepsilon)$. However, this implies $|x'-y'| > a_\varepsilon$. In this case we use $\omega = 1$ on $\mathbb{R}^{\theta-1}\setminus B(0,1)$, see the text above (5.47), to obtain

$$\omega(\tfrac{x'-y'}{a_\varepsilon}) - 1 = 0.$$

This shows that (5.59) vanishes for all $x', y' \in \mathbb{R}^{\theta-1}$ and $t, s \in (-1,1)$. □

Proposition 5.17. *Let ζ and κ be as in* (5.42), *Σ be as in* (5.43), *$\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, $d = \eta^2 - \tau^2$ satisfy*

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4},\tag{5.60}$$

q be as in (5.1), *$Q_{\eta,\tau}^{\zeta,\kappa}$ be as in* (5.18) *and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, there exists an $\varepsilon_{\mathrm{gr},3}$ in the interval $(0, \varepsilon_{ABC}]$, with $\varepsilon_{ABC} > 0$ chosen according to* (4.19), *such that the operator $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ has a right inverse which is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $\varepsilon \in (0, \varepsilon_{\mathrm{gr},3})$.*

Proof. The proof is split into four steps. In *Step 1* we define a first approximation for the right inverse of $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ denoted by R_ε. Moreover, in this step we also show that R_ε is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to ε. Then, in *Step 2* we calculate $(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon$. Afterwards, we find in *Step 3* that the product $(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon$ equals $I + K_\varepsilon + L_\varepsilon$, where K_ε and L_ε fulfil the inequalities

$$\|K_\varepsilon\|_{0\to1} \le C\frac{1+|\log(\varepsilon)|}{a_\varepsilon^2} \quad\text{and}\quad \|L_\varepsilon\|_{0\to0} \le Ca_\varepsilon(1+|\log(\varepsilon)|).\tag{5.61}$$

Based on these observations we use Proposition 2.30 in *Step 4* to prove the assertion.

Step 1. We define for $\varepsilon \in (0, \min\{\varepsilon_{ABC}, \varepsilon_{\mathrm{gr},2}\})$ with ε_{ABC} and $\varepsilon_{\mathrm{gr},2}$ chosen as in (4.19) and Proposition 5.15, respectively, $a_\varepsilon = \varepsilon^{1/6}$ and $n' \in \mathbb{Z}^{\theta-1}$

$$R_{n',\varepsilon} := (I + D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n') q)^{-1}$$

with $\zeta_{a_\varepsilon n'} = \zeta_{x_0'}$ as in (5.46) for $x_0' = a_\varepsilon n'$. These operators are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ with respect to $n' \in \mathbb{Z}^{\theta-1}$ and $\varepsilon \in (0, \min\{\varepsilon_{ABC}, \varepsilon_{\mathrm{gr},2}\})$ according to Proposition 5.15. Therefore, Proposition C.3 (see also (v) in Section 2.1) shows that

$$\begin{aligned} &R_\varepsilon : \mathcal{B}^0(\mathbb{R}^{\theta-1}) \to \mathcal{B}^0(\mathbb{R}^{\theta-1}), \\ &R_\varepsilon := \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} \vartheta_{n'}^{a_\varepsilon} \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}, \end{aligned}$$

is well-defined and uniformly bounded by

$$\|R_\varepsilon\|_{0\to0} \le 11^{\theta-1} \sup_{n' \in \mathbb{Z}^{\theta-1}} \left\|(I + D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n') q)^{-1}\right\|_{0\to0} \le C, \tag{5.62}$$

where $C > 0$ does not depend on ε. Furthermore, since $\vartheta_{n'}^{a_\varepsilon} \phi_{n'}^{a_\varepsilon} = \phi_{n'}^{a_\varepsilon}$ by construction, we have $R_\varepsilon = \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}$.

Step 2. Applying $I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q$ to R_ε yields

$$\begin{aligned} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q) R_\varepsilon &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q) \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \\ &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} \vartheta_{n'}^{a_\varepsilon} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q) \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \\ &\quad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} (1 - \vartheta_{n'}^{a_\varepsilon}) D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon}. \end{aligned}$$

Moreover, using Lemma 5.16 gives us

$$\begin{aligned} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q) R_\varepsilon &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} \vartheta_{n'}^{a_\varepsilon} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q) \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \\ &\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} (1 - \vartheta_{n'}^{a_\varepsilon}) E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \\ &= \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\mathrm{st.}} \vartheta_{n'}^{a_\varepsilon} (I + D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} q - E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q) \phi_{n'}^{a_\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \\ &\quad + E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q R_\varepsilon. \end{aligned}$$

Writing $D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q\phi_{n'}^{a_\varepsilon}$ as

$$\begin{aligned}
&D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q\phi_{n'}^{a_\varepsilon} + \big(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\big)q\phi_{n'}^{a_\varepsilon}\\
&\quad = \phi_{n'}^{a_\varepsilon} D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q + [D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z), \phi_{n'}^{a_\varepsilon}]Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q\\
&\qquad\qquad + \big(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\big)q\phi_{n'}^{a_\varepsilon}
\end{aligned}$$

and introducing the operators $L_{n',\varepsilon} := \big(D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')\big)q\phi_{n'}^{a_\varepsilon}$ and $K_{n',\varepsilon} := [D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z), \phi_{n'}^{a_\varepsilon}]Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q - E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}q\phi_{n'}^{a_\varepsilon}$ yields

$$\begin{aligned}
&(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon\\
&\quad = \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon}(I + D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n')q)R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon}\\
&\qquad + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(K_{n',\varepsilon} + L_{n',\varepsilon})R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon\\
&\quad = I + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}(K_{n',\varepsilon} + L_{n',\varepsilon})R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon,
\end{aligned} \tag{5.63}$$

where

$$\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon}\vartheta_{n'}^{a_\varepsilon} = \sum_{n'\in\mathbb{Z}^{\theta-1}} \vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon}\vartheta_{n'}^{a_\varepsilon} = \sum_{n'\in\mathbb{Z}^{\theta-1}} \phi_{n'}^{a_\varepsilon} = 1$$

was used.

Step 3. We start this step by setting

$$\begin{aligned}
K_\varepsilon &:= \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}K_{n',\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon} + E_\varepsilon(z)Q_{\eta,\tau}^{\zeta,\kappa}qR_\varepsilon,\\
L_\varepsilon &:= \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon}L_{n',\varepsilon}R_{n',\varepsilon}\vartheta_{n'}^{a_\varepsilon}.
\end{aligned}$$

Then, (5.63) shows $(I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)R_\varepsilon = I + K_\varepsilon + L_\varepsilon$. Since R_ε and $D_\varepsilon^{\zeta,\kappa}(z)$ are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, see *Step 1* and the text above (5.12), respectively, this implies that also $K_\varepsilon + L_\varepsilon$ is uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Moreover, Proposition C.3, Proposition 5.14, Proposition 5.15, (5.62) and $Q_{\eta,\tau}^{\zeta,\kappa} \in C_b^1(\mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$

imply

$$\begin{aligned}
\|K_\varepsilon\|_{0\to1} &\le \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} K_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to1} + \|\ E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q R_\varepsilon \|_{0\to1} \\
&\le \frac{C}{a_\varepsilon} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|K_{n',\varepsilon} R_{n',\varepsilon}\|_{0\to1} + \|\ E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q R_\varepsilon\|_{0\to1} \\
&\le \frac{C}{a_\varepsilon} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|[D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z), \phi_{n'}^{a_\varepsilon}] Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n') q - E_\varepsilon(z) Q_{\eta,\tau}^{\zeta,\kappa} q \phi_{n'}^{a_\varepsilon}\|_{0\to1} + \|\ E_\varepsilon(z)\|_{0\to1} \\
&\le \frac{C}{a_\varepsilon} \sup_{n'\in\mathbb{Z}^{\theta-1}} \Big(\|[D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z), \phi_{n'}^{a_\varepsilon}]\|_{0\to1} + \|E_\varepsilon(z)\|_{0\to1} \Big) + \|\ E_\varepsilon(z)\|_{0\to1} \\
&\le C \frac{1+|\log(\varepsilon)|}{a_\varepsilon} \Big(\sup_{n'\in\mathbb{Z}^{\theta-1}} \|\phi_{n'}^{a_\varepsilon}\|_{W^1_\infty(\mathbb{R}^{\theta-1})} + \frac{1}{a_\varepsilon} + 1 \Big) \\
&\le C \frac{1+|\log(\varepsilon)|}{a_\varepsilon} \Big(\frac{2}{a_\varepsilon} + 1 \Big) \\
&\le C \frac{1+|\log(\varepsilon)|}{a_\varepsilon^2}.
\end{aligned}$$

Similarly, we estimate L_ε with Proposition C.3, Proposition 5.14 and Proposition 5.15 by

$$\begin{aligned}
\|L_\varepsilon\|_{0\to0} &= \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} L_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to0} \\
&= \Big\| \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^{a_\varepsilon} \chi_{B(a_\varepsilon n', 3a_\varepsilon)} L_{n',\varepsilon} R_{n',\varepsilon} \vartheta_{n'}^{a_\varepsilon} \Big\|_{0\to0} \\
&\le C \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} L_{n',\varepsilon}\|_{0\to0} \\
&= C \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} \big(D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n') \big) q \phi_{n'}^{a_\varepsilon}\|_{0\to0} \\
&= C \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\chi_{B(a_\varepsilon n', 3a_\varepsilon)} \big(D_\varepsilon^{\zeta,\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa} - D_\varepsilon^{\zeta_{a_\varepsilon n'},\kappa}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n') \big) q \chi_{B(a_\varepsilon n', 3a_\varepsilon)} \phi_{n'}^{a_\varepsilon}\|_{0\to0} \\
&\le C a_\varepsilon (1 + |\log(\varepsilon)|).
\end{aligned}$$

This shows that (5.61) is valid and hence completes *Step 3.*

Step 4. Revisiting the considerations from the beginning of the current chapter shows that for $V = \eta I_N + \tau\beta$, $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ fulfilling (5.60), the following holds: $\cos\big(\frac{(\alpha\cdot\nu)V}{2}\big)^{-1} \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$,

$$\widetilde{V} = V \operatorname{sinc}\big(\tfrac{(\alpha\cdot\nu)V}{2}\big) \cos\big(\tfrac{(\alpha\cdot\nu)V}{2}\big)^{-1} = \widetilde{\eta} I_N + \widetilde{\tau}\beta \quad \text{with} \quad (\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\tfrac{\sqrt{d}}{2}\big)(\eta,\tau),$$

and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ fulfils $\inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma) - 4| > 0$. Thus, by Proposition 3.15 (iii) and Proposition 4.13 the operator $I + B_0(z)Vq$ is continuously invertible in $\mathcal{B}^0(\Sigma)$ and $\mathcal{B}^{1/2}(\Sigma)$. Hence, (5.9) and (5.11) imply that also $I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ is continuously invertible in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ and $\mathcal{B}^{1/2}(\mathbb{R}^{\theta-1})$. Applying Proposition 2.30 (for the choices $\mathcal{A} = I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$, $\mathcal{A}_0 = I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$, $\mathcal{T} = R_\varepsilon$, $\mathcal{K}_1 = 0$, $\mathcal{K}_2 = K_\varepsilon + L_\varepsilon$) shows that if the operator norm of

$$\begin{aligned}\widetilde{L}_\varepsilon &:= (D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon)\\ &\quad (= \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2)\end{aligned}$$

is bounded by $\frac{1}{2}$, then $I + D_\varepsilon^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q$ has a right inverse which is bounded by

$$2\big\|R_\varepsilon - (I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}(K_\varepsilon + L_\varepsilon)\big\|_{0\to 0}.$$

In particular, as $K_\varepsilon + L_\varepsilon$ and R_ε are uniformly bounded in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, this would yield the assertion. Using the estimates for L_ε and K_ε from *Step 3* as well as (5.12) and the text above it, we obtain for a fixed $r \in (0, \frac{1}{6})$

$$\begin{aligned}\|\widetilde{L}_\varepsilon\|_{0\to 0} &\leq \big\|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}K_\varepsilon\big\|_{0\to 0}\\ &\quad + \big\|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}L_\varepsilon\big\|_{0\to 0}\\ &\leq \big\|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\big\|_{1/2\to 0}\\ &\qquad\qquad \cdot \big\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\big\|_{1/2\to 1/2}\|K_\varepsilon\|_{0\to 1/2}\\ &\quad + \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{0\to 0}\\ &\qquad\qquad \cdot \big\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\big\|_{0\to 0}\|L_\varepsilon\|_{0\to 0}\\ &\leq \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{1/2\to 0}\\ &\qquad\qquad \cdot \big\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\big\|_{1/2\to 1/2}\|K_\varepsilon\|_{0\to 1}\\ &\quad + \|(D_0^{\zeta,\kappa}(z) - D_\varepsilon^{\zeta,\kappa}(z))Q_{\eta,\tau}^{\zeta,\kappa}q\|_{0\to 0}\\ &\qquad\qquad \cdot \big\|(I + D_0^{\zeta,\kappa}(z)Q_{\eta,\tau}^{\zeta,\kappa}q)^{-1}\big\|_{0\to 0}\|L_\varepsilon\|_{0\to 0}\\ &\leq C\Big(\frac{\varepsilon^{1/2-r}(1 + |\log(\varepsilon)|)}{a_\varepsilon^2} + a_\varepsilon(1 + |\log(\varepsilon)|)\Big)\\ &= C(1 + |\log(\varepsilon)|)(\varepsilon^{1/6-r} + \varepsilon^{1/6}) \leq C(1 + |\log(\varepsilon)|)\varepsilon^{1/6-r}.\end{aligned}$$

This shows that if we choose $\varepsilon_{\mathrm{gr},3} > 0$ sufficiently small, then $\|\widetilde{L}_\varepsilon\|_{0\to 0} < \frac{1}{2}$ for all $\varepsilon \in (0, \varepsilon_{\mathrm{gr},3})$. □

Finally, we are able to state the main result of Section 5.1 in the following proposition.

Proposition 5.18. *Let Σ be a rotated C_b^2-graph as described in the beginning of Section 5.1.2, $z \in \mathbb{C} \setminus \mathbb{R}$, q and $V = \eta I_N + \tau\beta$ be as in* (5.1) *and* (5.2), *$d = \eta^2 - \tau^2$ fulfil*

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4},$$

and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, there exists an $\varepsilon_{\mathrm{conv}} \in (0, \varepsilon_{ABC}]$, with $\varepsilon_{ABC} > 0$ chosen according to (4.19), *such that $I + B_\varepsilon(z)Vq$ has an inverse which is uniformly bounded in $\mathcal{B}^0(\Sigma)$ with respect to $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$.*

Proof. From the previous proposition, (5.10), (5.11) and (5.18) we directly get that the operator $I + \overline{B}_\varepsilon(z)Vq$ has a right inverse which is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_{\mathrm{gr},3})$. Using (5.8) shows that then $I + B_\varepsilon(z)Vq$ has a right inverse which is uniformly bounded for $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$ if $\varepsilon_{\mathrm{conv}} > 0$ is chosen small enough. Moreover, since $z \in \mathbb{C} \setminus \mathbb{R}$, Proposition 4.1 (i) implies that $I + B_\varepsilon(z)Vq$ has also a left inverse which yields that $I + B_\varepsilon(z)Vq$ is invertible and its inverse is uniformly bounded. $\square$

5.2 Main results

In this section we state and prove the main results of this chapter. After dealing with the cases where Σ is a hyperplane in Section 5.1.1 and a rotated graph of a C_b^2-function in Section 5.1.2, we return to our general assumption that $\Sigma \subset \mathbb{R}^\theta$ is a special C^2-surface as in Definition 2.1. We start by providing a useful lemma regarding Dirac operators with δ-shell potentials.

Lemma 5.19. *Let $O_1, O_2 \subset \mathbb{R}^\theta$ be open sets such that their boundaries $S_1 = \partial O_1$, $S_2 = \partial O_2$ are special C^2-surfaces as in Definition 2.1, $\widehat{V}_1 = \widehat{V}_1^* \in W^1_\infty(S_1; \mathbb{C}^{N\times N})$, $\widehat{V}_2 = \widehat{V}_2^* \in W^1_\infty(S_2; \mathbb{C}^{N\times N})$, $\varphi \in C_b^1(\mathbb{R}^\theta)$ and $H_{\widehat{V}_1\delta_{S_1}}$, $H_{\widehat{V}_2\delta_{S_2}}$ be Dirac operators with δ-shell potentials as in Definition 3.12. Moreover, assume that there exists an open set $O \subset \mathbb{R}^\theta$ such that $\operatorname{supp}\varphi \subset O$ and $O_1 \cap O = O_2 \cap O$, and $\widehat{V}_1 = \widehat{V}_2$ σ-a.e on $S_1 \cap O (= S_2 \cap O)$. Then, $H_{\widehat{V}_1\delta_{S_1}}\varphi = H_{\widehat{V}_2\delta_{S_2}}\varphi$, where φ is viewed as an multiplication operator in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, and for $u \in \operatorname{dom} H_{\widehat{V}_j\delta_{S_j}}$, $j \in \{1,2\}$, one has $u \in \operatorname{dom} H_{\widehat{V}_j\delta_{S_j}}\varphi$ and*

$$\varphi H_{\widehat{V}_j\delta_{S_j}} u = H_{\widehat{V}_j\delta_{S_j}} \varphi u + i(\alpha \cdot \nabla\varphi)u.$$

Proof. In this proof we use the notations $O_{j,+} = O_j$ and $O_{j,-} = \mathbb{R}^\theta \setminus \overline{O_j}$ for $j \in \{1,2\}$. Moreover, ν_j denotes the unit normal vector field on S_j pointing outwards of $O_{j,+}$. Now, let us start by showing $H_{\widehat{V}_1\delta_{S_1}}\varphi = H_{\widehat{V}_2\delta_{S_2}}\varphi$. To do so, let $u \in \operatorname{dom} H_{\widehat{V}_1\delta_{S_1}}\varphi$. Then, $\varphi u \in \operatorname{dom} H_{\widehat{V}_1\delta_{S_1}} \subset H^1(\mathbb{R}^\theta \setminus S_1)$. In particular, $(\varphi u) \upharpoonright O_{1,\pm} \in H^1(O_{1,\pm}; \mathbb{C}^N)$.

Furthermore, $(\varphi u) \upharpoonright O_{2,\pm} \in H^1(O_{2,\pm}; \mathbb{C}^N)$ since $\operatorname{supp} \varphi \subset O$ and $O_{1,\pm} \cap O = O_{2,\pm} \cap O$. Thus, we can apply the trace operator to $(\varphi u) \upharpoonright O_{2,\pm}$ and obtain

$$\begin{aligned}
&\Big(i(\alpha \cdot \nu_2)(\boldsymbol{t}_{S_2}^+ - \boldsymbol{t}_{S_2}^-) + \frac{\widehat{V}_2}{2}(\boldsymbol{t}_{S_2}^+ + \boldsymbol{t}_{S_2}^-)\Big)\varphi u \\
&= \begin{cases} \Big(i(\alpha \cdot \nu_2)(\boldsymbol{t}_{S_2}^+ - \boldsymbol{t}_{S_2}^-) + \frac{\widehat{V}_2}{2}(\boldsymbol{t}_{S_2}^+ + \boldsymbol{t}_{S_2}^-)\Big)\varphi u & \text{on } S_2 \cap O, \\ 0 & \text{on } S_2 \setminus O, \end{cases} \\
&= \begin{cases} \Big(i(\alpha \cdot \nu_1)(\boldsymbol{t}_{S_1}^+ - \boldsymbol{t}_{S_1}^-) + \frac{\widehat{V}_1}{2}(\boldsymbol{t}_{S_1}^+ + \boldsymbol{t}_{S_1}^-)\Big)\varphi u & \text{on } S_1 \cap O, \\ 0 & \text{on } S_1 \setminus O, \end{cases} \\
&= 0,
\end{aligned}$$

where we used $\varphi u \in \operatorname{dom} H_{\widehat{V}_1 \delta_{S_1}}$, $\operatorname{supp} \varphi \subset O$ and $S_1 \cap O = S_2 \cap O$. Hence, $u\varphi$ fulfils the boundary condition

$$\Big(i(\alpha \cdot \nu_2)(\boldsymbol{t}_{S_2}^+ - \boldsymbol{t}_{S_2}^-) + \frac{\widehat{V}_2}{2}(\boldsymbol{t}_{S_2}^+ + \boldsymbol{t}_{S_2}^-)\Big)\varphi u = 0,$$

i.e. $\varphi u \in \operatorname{dom} H_{\widehat{V}_2 \delta_{S_2}}$. This shows $\operatorname{dom} H_{\widehat{V}_1 \delta_{S_1}} \varphi \subset \operatorname{dom} H_{\widehat{V}_2 \delta_{S_2}} \varphi$. The reverse inclusion can be proven in the same way. Moreover, for $u \in \operatorname{dom} H_{\widehat{V}_1 \delta_{S_1}} \varphi = \operatorname{dom} H_{\widehat{V}_2 \delta_{S_2}} \varphi$ the equality

$$\begin{aligned}
H_{\widehat{V}_1 \delta_{S_1}} \varphi u &= \begin{cases} (-i(\alpha \cdot \nabla) + m\beta)(\varphi u) \upharpoonright O_{1,\pm} \cap O & \text{in } O_{1,\pm} \cap O, \\ 0 & \text{else}, \end{cases} \\
&= \begin{cases} (-i(\alpha \cdot \nabla) + m\beta)(\varphi u) \upharpoonright O_{2,\pm} \cap O & \text{in } O_{2,\pm} \cap O, \\ 0 & \text{else}, \end{cases} \\
&= H_{\widehat{V}_2 \delta_{S_2}} \varphi u
\end{aligned}$$

is valid, showing $H_{\widehat{V}_1 \delta_{S_1}} \varphi = H_{\widehat{V}_2 \delta_{S_2}} \varphi$. With similar arguments as above one proves $\varphi u \in \operatorname{dom} H_{\widehat{V}_j \delta_{S_j}}$, $j \in \{1, 2\}$, for $u \in \operatorname{dom} H_{\widehat{V}_j \delta_{S_j}}$. Moreover, the product rule gives us for $u \in \operatorname{dom} H_{\widehat{V}_j \delta_{S_j}}$

$$\varphi H_{\widehat{V}_j \delta_{S_j}} u = H_{\widehat{V}_j \delta_{S_j}} \varphi u + i(\alpha \cdot \nabla \varphi) u. \qquad \square$$

The upcoming theorem, which is one of the main results of this thesis, shows that for electrostatic and Lorentz scalar interactions, i.e. $V = \eta I_N + \tau\beta$, $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, the simple condition

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}, \qquad d = \eta^2 - \tau^2, \tag{5.64}$$

guarantees the norm resolvent convergence of H_{V_ε} for $\varepsilon \to 0$.

Theorem 5.20. *Let q be as in (5.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ satisfy (5.64), V_ε be defined by (4.3) and $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, set $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ with $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$ where $d = \eta^2 - \tau^2$. Then, the operator $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and there exists an $\varepsilon_{\mathrm{conv}} > 0$ such that for any $r \in (0, \frac{1}{2})$ exists a $C > 0$ such that*

$$\big\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$. In particular, H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Proof. First, we observe that (5.64), $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4\tan\big(\frac{\sqrt{d}}{2}\big)^2$ imply

$$\inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma) - 4| > 0$$

and thus $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint by Proposition 3.15. Moreover, we note that if Σ is a rotated C_b^2-graph, then the assertion follows directly from Theorem 4.15, the text in the beginning of Chapter 5 and Proposition 5.18.

In the general setting $\Sigma \subset \mathbb{R}^\theta$ is a special C^2-surface as in Definition 2.1. The surface Σ is in this case a subset of $\bigcup_{l=1}^p \Sigma_l$ with $\Sigma_l = \{\kappa_l(x', \zeta_l(x')) : x' \in \mathbb{R}^{\theta-1}\}$ for $l \in \{1, \dots, p\}$. Thus, we prove the general case by reducing it to the case of rotated C_b^2-graphs. However, V is only defined on Σ and therefore in general only on parts of Σ_l, $l \in \{1, \dots, p\}$. Hence, we define suitable extensions of V to Σ_l in the following way: Let $\widehat{V}$, $\widehat{\eta}$, $\widehat{\tau}$ be the C_b^1-extensions of V, τ and η defined by (4.10). Moreover, we set $V_l := \widehat{V} \upharpoonright \Sigma_l$, $\eta_l := \widehat{\eta} \upharpoonright \Sigma_l$, $\tau_l := \widehat{\tau} \upharpoonright \Sigma_l$ and $d_l := \eta_l^2 - \tau_l^2$ for $l \in \{1, \dots, p\}$. Then, these functions also satisfy

$$\sup_{x_{\Sigma_l} \in \Sigma_l} d_l(x_{\Sigma_l}) < \frac{\pi^2}{4} \qquad \forall l \in \{1, \dots, p\} \tag{5.65}$$

by construction as $d = \eta^2 - \tau^2$ satisfies (5.64).

In order to be able to reduce the general case, we choose the C^1-partition of unity $\widehat{\varphi}_1, \dots, \widehat{\varphi}_p \in C_b^1(\mathbb{R}^\theta)$ for $\Omega_{\frac{\varepsilon_{\mathrm{tub}}}{2}}$ from Corollary A.5, which fulfils $\operatorname{supp}\widehat{\varphi}_l \cap \Sigma \subset W_l$, where $W_1, \dots, W_p$ is the open cover of Σ from Definition 2.1. Moreover, let for $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$ $\Omega_{l,\varepsilon}$ be the tubular neighbourhood of Σ_l and $V_{l,\varepsilon}$ be defined analogously to V_ε in (4.3). We claim that $\Omega_\varepsilon \cap \operatorname{supp}\widehat{\varphi}_l = \Omega_{\varepsilon,l} \cap \operatorname{supp}\widehat{\varphi}_l$ for all $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$ and $l \in \{1, \dots p\}$. Indeed, if $x \in \Omega_\varepsilon \cap \operatorname{supp}\widehat{\varphi}_l$, then there exists $(x_\Sigma, t) \in \Sigma \times (-\varepsilon, \varepsilon)$ such that $x = x_\Sigma + t\nu(x_\Sigma)$. The equation $0 \neq \widehat{\varphi}_l(x) = \widehat{\varphi}_l(x_\Sigma)\varpi(t)$, see Corollary A.5, implies $x_\Sigma \in \operatorname{supp}\widehat{\varphi}_l \cap \Sigma \subset W_l \cap \Sigma = W_l \cap \Sigma_l$. Thus, $x_\Sigma \in \Sigma_l$ and $\nu(x_\Sigma) = \nu_l(x_\Sigma)$. Consequently, $x = x_\Sigma + t\nu_l(x_\Sigma) \in \Omega_{l,\varepsilon} \cap \operatorname{supp}\widehat{\varphi}_l$ and additionally

$$V_\varepsilon(x) = V(x_\Sigma)\frac{q\big(\frac{t}{\varepsilon}\big)}{\varepsilon} = V_l(x_\Sigma)\frac{q\big(\frac{t}{\varepsilon}\big)}{\varepsilon} = V_{l,\varepsilon}(x).$$

The reverse inclusion can be shown in exactly the same way. This implies in particular $V_{l,\varepsilon}\widehat{\varphi}_l = V_\varepsilon \widehat{\varphi}_l$ for $l \in \{1,\dots,p\}$ and $\varepsilon \in (0, \varepsilon_{\text{tub}})$. Hence, we get for $l \in \{1,\dots,p\}$ and $\varepsilon \in (0, \varepsilon_{\text{tub}})$ the identity

$$H_{V_{l,\varepsilon}}\widehat{\varphi}_l u = H_{V_\varepsilon}\widehat{\varphi}_l u \quad \forall u \in \operatorname{dom} H_{V_\varepsilon} = \operatorname{dom} H_{V_{l,\varepsilon}} = H^1(\mathbb{R}^\theta;\mathbb{C}^N),$$

where $H_{V_{l,\varepsilon}} = H + V_{l,\varepsilon}$ and $H_{V_\varepsilon} = H + V_\varepsilon$ with H being the free Dirac operator; cf. (4.4) and Definition 3.2. Additionally to $\widehat{\varphi}_1,\dots,\widehat{\varphi}_p$, we introduce $\widehat{\varphi}_{p+1} := 1 - \sum_{l=1}^p \widehat{\varphi}_l$. The fact that $\widehat{\varphi}_1,\dots,\widehat{\varphi}_p$ is a partition of unity for $\Omega_{\frac{\varepsilon_{\text{tub}}}{2}}$ shows that for all $\varepsilon \in (0, \frac{\varepsilon_{\text{tub}}}{2})$ $V_\varepsilon \widehat{\varphi}_{p+1} = 0$ and thus $H_{V_\varepsilon}\widehat{\varphi}_{p+1} u = H\widehat{\varphi}_{p+1} u$ for $u \in H^1(\mathbb{R}^\theta;\mathbb{C}^N)$. These observations, setting $H_{V_{p+1,\varepsilon}} = H$ and applying the product rule yields for $l \in \{1,\dots,p+1\}$, $\varepsilon \in (0, \frac{\varepsilon_{\text{tub}}}{2})$ and $u \in \operatorname{dom} H_{V_\varepsilon} = H^1(\mathbb{R}^\theta;\mathbb{C}^N)$

$$\widehat{\varphi}_l H_{V_\varepsilon} u = H_{V_\varepsilon}\widehat{\varphi}_l u + i(\alpha \cdot \nabla\widehat{\varphi}_l)u = H_{V_{l,\varepsilon}}\widehat{\varphi}_l u + i(\alpha \cdot \nabla\widehat{\varphi}_l). \tag{5.66}$$

Next, we construct a resolvent for H_{V_ε} in terms of the operators $H_{V_{l,\varepsilon}}$. We use (5.66) to get for $u \in \operatorname{dom} H_{V_\varepsilon}$ and $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}
\Big(\sum_{l=1}^{p+1} &(H_{V_{l,\varepsilon}} - z)^{-1}\widehat{\varphi}_l\Big)(H_{V_\varepsilon} - z)u \\
&= \sum_{l=1}^{p+1} (H_{V_{l,\varepsilon}} - z)^{-1}(H_{V_{l,\varepsilon}} - z)\widehat{\varphi}_l u + i(H_{V_{l,\varepsilon}} - z)^{-1}(\alpha \cdot \nabla\widehat{\varphi}_l)u \\
&= \sum_{l=1}^{p+1} \widehat{\varphi}_l u + i(H_{V_{l,\varepsilon}} - z)^{-1}(\alpha \cdot \nabla\widehat{\varphi}_l)u \\
&= \Big(I + \sum_{l=1}^{p+1} i(H_{V_{l,\varepsilon}} - z)^{-1}(\alpha \cdot \nabla\widehat{\varphi}_l)\Big)u.
\end{aligned} \tag{5.67}$$

In particular, if $|\operatorname{Im} z| > \sum_{l=1}^{p+1} \|\alpha \cdot \nabla\widehat{\varphi}_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, then

$$I + \sum_{l=1}^{p+1} i(H_{V_{l,\varepsilon}} - z)^{-1}(\alpha \cdot \nabla\widehat{\varphi}_l)$$

is continuously invertible in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and

$$(H_{V_\varepsilon} - z)^{-1} = \Big(I + \sum_{l=1}^{p+1} i(H_{V_{l,\varepsilon}} - z)^{-1}(\alpha \cdot \nabla\widehat{\varphi}_l)\Big)^{-1}\Big(\sum_{l=1}^{p+1}(H_{V_{l,\varepsilon}} - z)^{-1}\widehat{\varphi}_l\Big) \tag{5.68}$$

for all $\varepsilon \in (0, \frac{\varepsilon_{\text{tub}}}{2})$. Next, we find a similar resolvent formula for the resolvent of $H_{\widetilde{V}\delta_\Sigma}$. Again, it is important to establish relations between $H_{\widetilde{V}\delta_\Sigma}$ and $H_{\widetilde{V}_l\delta_{\Sigma_l}}$.

Here, $\widetilde{V} = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)V$, $\widetilde{V}_l = \operatorname{tanc}\big(\frac{\sqrt{d_l}}{2}\big)V_l$, $l \in \{1,\dots,p\}$, $H_{\widetilde{V}\delta_\Sigma}$ is defined as in Definition 3.12 and $H_{\widetilde{V}_l\delta_{\Sigma_l}}$ is defined in the same way as $H_{\widetilde{V}\delta_\Sigma}$ (with Σ_l instead of Σ and $\widetilde{V}_l$ instead of V). Furthermore, we set $\widetilde{V}_{p+1} = 0$ and $\Sigma_{p+1} = \Sigma$. In this case $H_{\widetilde{V}_{p+1}\delta_{\Sigma_{p+1}}}$ coincides with the free Dirac operator H. Note that we already know from the comments at the beginning of the proof that (5.64) and (5.65) imply that $H_{\widetilde{V}\delta_\Sigma}$ and $H_{\widetilde{V}_l\delta_{\Sigma_l}}$, $l \in \{1,\dots,p\}$, are self-adjoint in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. Moreover, Lemma 5.19 shows that if $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$, then $\widehat{\varphi}_l u \in \operatorname{dom} H_{\widetilde{V}_l\delta_{\Sigma_l}}$ and

$$\widehat{\varphi}_l H_{\widetilde{V}\delta_\Sigma} u = H_{\widetilde{V}\delta_\Sigma}\widehat{\varphi}_l u + i(\alpha\cdot\nabla\widehat{\varphi}_l)u = H_{\widetilde{V}_l\delta_{\Sigma_l}}\widehat{\varphi}_l u + i(\alpha\cdot\nabla\widehat{\varphi}_l)u \tag{5.69}$$

for all $l \in \{1,\dots,p+1\}$. Using (5.69) one argues with the same steps as in (5.67) that for $u \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ and $z \in \mathbb{C}\setminus\mathbb{R}$

$$\Big(\sum_{l=1}^{p+1}(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\widehat{\varphi}_l\Big)(H_{\widetilde{V}\delta_\Sigma} - z)u = \Big(I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\widehat{\varphi}_l)\Big)u.$$

Hence, if $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\widehat{\varphi}_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, then $I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\widehat{\varphi}_l)$ is continuously invertible in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and

$$(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} = \Big(I + \sum_{l=1}^{p+1} i(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}(\alpha\cdot\nabla\widehat{\varphi}_l)\Big)^{-1}\Big(\sum_{l=1}^{p+1}(H_{\widetilde{V}_l\delta_{\Sigma_l}} - z)^{-1}\widehat{\varphi}_l\Big). \tag{5.70}$$

Thus, if $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\widehat{\varphi}_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$, then the assertion follows from comparing (5.68) and (5.70) with one another, and the case of the rotated C_b^2-graph. Moreover, we can get rid of the assumption $|\operatorname{Im} z| > \sum_{l=1}^{p+1}\|\alpha\cdot\nabla\widehat{\varphi}_l\|_{L^\infty(\mathbb{R}^\theta;\mathbb{C}^{N\times N})}$ by applying the resolvent identity

$$\begin{aligned}&(H_{\widetilde{V}\delta_\Sigma} - w)^{-1} - (H_{V_\varepsilon} - w)^{-1} = (I + (w-z)(H_{\widetilde{V}\delta_\Sigma} - w)^{-1})\\ &\cdot\big((H_{\widetilde{V}\delta_\Sigma} - z)^{-1} - (H_{V_\varepsilon} - z)^{-1}\big)(I + (w-z)(H_{V_\varepsilon} - w)^{-1}) \quad \forall z,w \in \mathbb{C}\setminus\mathbb{R}.\end{aligned}$$

This completes the proof of Theorem 5.20. □

In the next theorem we add a magnetic term to V and show that then H_{V_ε} also converges to a Dirac operator with δ-shell potential with electrostatic and Lorentz scalar interactions. However, in the following theorem the rescaling of the interaction strengths is different.

Theorem 5.21. *Let q be as in* (5.1), *$V = \eta I_N + \tau\beta + \pi(\alpha\cdot\nu)$ with $\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$ and the unit normal vector ν on Σ, $d = \eta^2 - \tau^2$ satisfy*

$$\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4} \quad \text{and} \quad \inf_{x_\Sigma\in\Sigma}|d(x_\Sigma)| > 0,$$

V_ε be defined by (4.3) *and $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, set*

$$(\widetilde{\eta}, \widetilde{\tau}) := \frac{-2}{\sqrt{d}\tan\left(\frac{\sqrt{d}}{2}\right)}(\eta, \tau)$$

and $\widetilde{V} := \widetilde{\eta} I_N + \widetilde{\tau}\beta$. Then, the operator $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and there exists an $\varepsilon_{\mathrm{conv}} > 0$ such that for any $r \in (0, \frac{1}{2})$ exists a $C > 0$ such that

$$\left\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\right\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$. In particular, H_{V_ε} converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Proof. We start by calculating

$$\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = \frac{4(\eta^2 - \tau^2)}{d\tan\left(\frac{\sqrt{d}}{2}\right)^2} = \frac{4}{\tan\left(\frac{\sqrt{d}}{2}\right)^2}. \tag{5.71}$$

Thus, the assumptions regarding d imply that the $\inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma) - 4| > 0$ and hence $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint according to Proposition 3.15.

For the remaining proof we use an idea from [24, Section 8]. This idea makes use of the fact stated in Proposition 3.15 (i) that the two Dirac operators $H_{\widetilde{V}\delta_\Sigma}$ and $H_{-4(\widetilde{V}/\widetilde{d})\delta_\Sigma}$ are unitarily equivalent. More precisely, $H_{\widetilde{V}\delta_\Sigma} = UH_{-4(\widetilde{V}/\widetilde{d})\delta_\Sigma}U$, where U is the self-adjoint unitary multiplication operator in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ which is induced by the function $w = \chi_{\Omega_+} - \chi_{\Omega_-}$. If we set $F := \eta I_N + \tau\beta$ and $\widetilde{F} := \mathrm{tanc}\left(\frac{\sqrt{d}}{2}\right)F$, we get in the current setting

$$\widetilde{V} = \frac{-2}{\sqrt{d}\tan\left(\frac{\sqrt{d}}{2}\right)}F$$

and by (5.71)

$$\frac{-4}{\widetilde{d}}\widetilde{V} = \frac{2\tan\left(\frac{\sqrt{d}}{2}\right)}{\sqrt{d}}F = \mathrm{tanc}\left(\tfrac{\sqrt{d}}{2}\right)F = \widetilde{F}.$$

In particular,

$$UH_{\widetilde{F}\delta_\Sigma}U = H_{\widetilde{V}\delta_\Sigma}.$$

Note also that Theorem 5.20 shows that H_{F_ε}, where F_ε is analogously defined as V_ε in (4.3) with F instead of V, converges to $H_{\widetilde{F}\delta_\Sigma}$ in the norm resolvent sense. In this proof we find unitary multiplication operators W_ε such that

$$W_\varepsilon^* H_{F_\varepsilon} W_\varepsilon = H_{V_\varepsilon}$$

and $W_\varepsilon \to U$ for $\varepsilon \to 0$ in a suitable sense. Furthermore, using the convergence properties of these operators and H_{F_ε} we show that H_{V_ε} converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$. Having explained the ideas, we start the main part of the proof by defining for $\varepsilon \in (0, \varepsilon_{\text{tub}})$ the function

$$w_\varepsilon : \mathbb{R}^\theta \to \mathbb{C}, \qquad w_\varepsilon(x) := \begin{cases} 1, & x \in \Omega_+ \setminus \Omega_\varepsilon, \\ e^{i\pi \int_{-1}^{t/\varepsilon} q(s)\,ds}, & x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_\varepsilon, \\ -1, & x \in \Omega_- \setminus \Omega_\varepsilon. \end{cases}$$

This function is well-defined according to Proposition 2.12. Moreover, $\int_{-1}^{1} q(s)\,ds = 1$ shows that w_ε is continuous. We define W_ε to be the unitary multiplication operator in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ induced by w_ε. Next, we show $w_\varepsilon \in W^1_\infty(\mathbb{R}^\theta)$. To do so, let us fix $x = \iota(x_\Sigma, t) \in \Omega_\varepsilon$ and $x' \in \mathbb{R}^{\theta-1}$ such that $\varkappa_l(x') = x_\Sigma$ for a $l \in \{1, \dots, p\}$. This implies $x = \iota_l(x', t)$ with ι_l from Definition 2.7. In the proof of Lemma 4.3 we showed that $\iota_l : \mathbb{R}^{\theta-1} \times \mathbb{R} \to \mathbb{R}^\theta$ is locally around (x', t) a diffeomorphism. Moreover, using $w_\varepsilon \circ \iota_l(x', t) = e^{i\pi \int_{-1}^{t/\varepsilon} q(s)\,ds}$, the chain rule and formula (4.12) for $(D\iota_l)(x', t)$ yields

$$\begin{aligned} \nabla w_\varepsilon(x) &= \big(((D\iota_l)(x', t))^{-1}\big)^T \nabla(w_\varepsilon \circ \iota_l)(x', t) \\ &= \big(\; * \mid \nu_l(x') \;\big) \frac{i\pi q\big(\frac{t}{\varepsilon}\big)}{\varepsilon} e^{i\pi \int_{-1}^{t/\varepsilon} q(s)\,ds} e_\theta \\ &= \nu(x_\Sigma) \frac{i\pi q\big(\frac{t}{\varepsilon}\big)}{\varepsilon} w_\varepsilon(x), \end{aligned}$$

where e_θ is the θ-th Euclidean unit vector in $\mathbb{R}^\theta$. Hence,

$$\nabla w_\varepsilon(x) = \begin{cases} \nu(x_\Sigma) \frac{i\pi q\left(\frac{t}{\varepsilon}\right)}{\varepsilon} w_\varepsilon(x), & x = \iota(x_\Sigma, t) \in \Omega_\varepsilon, \\ 0, & x \notin \Omega_\varepsilon, \end{cases}$$

and $w_\varepsilon \in W^1_\infty(\mathbb{R}^\theta)$. These considerations, $\frac{1}{w_\varepsilon} = \overline{w_\varepsilon} \in W^1_\infty(\mathbb{R}^\theta)$, and the definition of V_ε in (4.3) show $\operatorname{dom} W_\varepsilon^* H_{F_\varepsilon} W_\varepsilon = \operatorname{dom} H_{V_\varepsilon} = H^1(\mathbb{R}^\theta; \mathbb{C}^N)$ and

$$W_\varepsilon^* H_{F_\varepsilon} W_\varepsilon = H_{F_\varepsilon} - i\overline{w_\varepsilon}(\alpha \cdot \nabla w_\varepsilon) = H_{V_\varepsilon};$$

cf. [24, Section 8, below the proof of Theorem 2.6]. We note that w_ε converges pointwise to $w = \chi_{\Omega_+} - \chi_{\Omega_-}$ and therefore W_ε converges in the strong sense to the operator U. In addition, for $\varepsilon \in (0, \varepsilon_{\text{tub}})$ the estimate

$$\|(W_\varepsilon^* - U)u\|_{L^2(\mathbb{R}^\theta; \mathbb{C}^N)} \le \varepsilon^{1/2} C \|u\|_{H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)} \quad \forall u \in H^1(\mathbb{R}^\theta \setminus \Sigma) \tag{5.72}$$

is also valid. We postpone the verification of this fact to the end of the proof. Moreover, $(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}$ also acts as a bounded operator from $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$. Indeed, one can show this by using $\operatorname{dom} H_{\widetilde{F}\delta_\Sigma} \subset H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$, see Definition 3.12, the boundedness of $(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}$ acting as an operator in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$, the continuous embedding of $H^1(\mathbb{R}^\theta \setminus \Sigma; \mathbb{C}^N)$ in $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and the closed graph theorem. Thus, (5.72) gives us for $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$

$$\big\|(W_\varepsilon^* - U)(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2}.$$

This observation, the norm resolvent convergence of H_{F_ε} (see Theorem 5.20) and the fact that U and W_ε are unitary operators let us estimate for $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$ (with $\varepsilon_{\mathrm{conv}} > 0$ from Theorem 5.20)

$$\begin{aligned}
\big\|(H_{V_\varepsilon} - z)^{-1} - &(H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= \|W_\varepsilon^*(H_{F_\varepsilon} - z)^{-1}W_\varepsilon - U(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}U\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\leq \big\|W_\varepsilon^*((H_{F_\varepsilon} - z)^{-1} - (H_{\widetilde{F}\delta_\Sigma} - z)^{-1})W_\varepsilon\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|(W_\varepsilon^* - U)(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}W_\varepsilon\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|U(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}(W_\varepsilon - U)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= \big\|(H_{F_\varepsilon} - z)^{-1} - (H_{\widetilde{F}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|(W_\varepsilon^* - U)(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}(W_\varepsilon - U)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&= \big\|(H_{F_\varepsilon} - z)^{-1} - (H_{\widetilde{F}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|(W_\varepsilon^* - U)(H_{\widetilde{F}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\
&\quad + \big\|(W_\varepsilon^* - U)(H_{\widetilde{F}\delta_\Sigma} - \overline{z})^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}. \\
&\leq C\varepsilon^{1/2-r} + C\varepsilon^{1/2} \\
&\leq C\varepsilon^{1/2-r},
\end{aligned}$$

where $r \in (0, \frac{1}{2})$. Hence, it only remains to prove (5.72).

We start by choosing $u = u_+ \oplus u_-$ with $u_\pm \in C_0^\infty(\overline{\Omega_\pm}; \mathbb{C}^N)$ and get for $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$

$$\begin{aligned}
\|(W_\varepsilon^* - U)u\|^2_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)} &= \int_{\Omega_\varepsilon} |(\overline{w_\varepsilon(x)} - \chi_{\Omega_+}(x) + \chi_{\Omega_-}(x))u(x)|^2\,dx \\
&\leq 4\int_{\Omega_\varepsilon} |u(x)|^2\,dx.
\end{aligned}$$

Moreover, by Corollary 2.10 (where we use $\varepsilon_{\mathrm{tub}} < \varepsilon_\iota$) and Proposition 2.12 we obtain

$$\begin{aligned}
\int_{\Omega_\varepsilon} |u(x)|^2\,dx \leq& C\int_{-\varepsilon}^{\varepsilon}\int_\Sigma |u(x_\Sigma + t\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt \\
=& C\varepsilon\Big(\int_{-1}^{0}\int_\Sigma |u_+(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt \\
&+ \int_0^1\int_\Sigma |u_-(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt\Big).
\end{aligned}$$

Next, we estimate the term $\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt$. Note that the smoothness of u implies that for $t \in (-1,0)$ the function

$$\Sigma \ni x_\Sigma \mapsto u_+(x_\Sigma + t\varepsilon\nu(x_\Sigma))$$

coincides with the trace of the function $\tau_{\varepsilon t}^{\Omega_+}u$, where $\tau_{\varepsilon t}^{\Omega_\pm}$ is the shift operator introduced in (4.18) (for $\delta = \varepsilon t$). Thus,

$$\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma + t\varepsilon\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt = \int_{-1}^{0}\int_{\Sigma}|\boldsymbol{t}_\Sigma\tau_{\varepsilon t}^{\Omega_+}u(x_\Sigma)|^2\,d\sigma(x_\Sigma)\,dt.$$

Now, Proposition 2.3 and Corollary 4.5 let us estimate this term in the following way:

$$\begin{aligned}\int_{-1}^{0}\int_{\Sigma}|\boldsymbol{t}_\Sigma\tau_{\varepsilon t}^{\Omega_+}u_+(x_\Sigma)|^2\,d\sigma(x_\Sigma)\,dt &= \int_{-1}^{0}\|\boldsymbol{t}_\Sigma\tau_{\varepsilon t}^{\Omega_+}u_+\|^2_{L^2(\Sigma;\mathbb{C}^N)}\,dt\\ &\le \int_{-1}^{0}\|\boldsymbol{t}_\Sigma\tau_{\varepsilon t}^{\Omega_+}u_+\|^2_{H^{1/2}(\Sigma;\mathbb{C}^N)}\,dt\\ &\le C\int_{-1}^{0}\|\tau_{\varepsilon t}^{\Omega_+}u_+\|^2_{H^1(\Omega_+;\mathbb{C}^N)}\,dt\\ &\le C\|u_+\|^2_{H^1(\Omega_+;\mathbb{C}^N)}.\end{aligned}$$

Therefore,

$$\int_{-1}^{0}\int_{\Sigma}|u_+(x_\Sigma + t\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt \le C\|u_+\|^2_{H^1(\Omega_+;\mathbb{C}^N)};$$

in the same way one gets

$$\int_{0}^{1}\int_{\Sigma}|u_-(x_\Sigma + t\nu(x_\Sigma))|^2\,d\sigma(x_\Sigma)\,dt \le C\|u_-\|^2_{H^1(\Omega_-;\mathbb{C}^N)}.$$

This implies (5.72) for $u \in C_0^\infty(\overline{\Omega_+};\mathbb{C}^N) \oplus C_0^\infty(\overline{\Omega_-};\mathbb{C}^N)$, which is a dense subspace of $H^1(\mathbb{R}^\theta \setminus \Sigma;\mathbb{C}^N) = H^1(\Omega_+;\mathbb{C}^N) \oplus H^1(\Omega_-;\mathbb{C}^N)$; see e.g. [54, Chapter 3]. Hence, (5.72) is valid. □

An immediate consequence of the two previous theorems is the following corollary.

Corollary 5.22. *Let q be as in (5.1), $\widetilde{V} = \widetilde{\eta}I_N + \widetilde{\tau}\beta$ with $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$, $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$ satisfy either*

$$\sup_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| < 4 \quad \text{or} \quad \inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| > 4,$$

and $z \in \mathbb{C} \setminus \mathbb{R}$. *Moreover, let the interaction strengths* $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ *be given by*

$$(\eta,\tau) := \begin{cases} \frac{2\arctan\left(\frac{\sqrt{\widetilde{d}}}{2}\right)}{\sqrt{\widetilde{d}}}(\widetilde{\eta},\widetilde{\tau}), & \sup_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| < 4, \\ \frac{-2\arctan\left(\frac{2}{\sqrt{\widetilde{d}}}\right)}{\sqrt{\widetilde{d}}}(\widetilde{\eta},\widetilde{\tau}), & \inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| > 4, \end{cases}$$
$$V := \begin{cases} \eta I_N + \tau\beta, & \sup_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| < 4, \\ \eta I_N + \tau\beta + \pi(\alpha\cdot\nu), & \inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| > 4, \end{cases}$$

and V_ε *be defined by* (4.3). *Then, the operator* $H_{\widetilde{V}\delta_\Sigma}$ *is self-adjoint and there exists an* $\varepsilon_{\mathrm{conv}} > 0$ *such that for any* $r \in (0, \frac{1}{2})$ *exists a* $C > 0$ *such that*

$$\left\|(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\right\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\varepsilon^{1/2-r}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conv}})$. *In particular,* H_{V_ε} *converges to* $H_{\widetilde{V}\delta_\Sigma}$ *in the norm resolvent sense as* $\varepsilon \to 0$.

Proof. By definition, we have

$$d = \eta^2 - \tau^2 = \begin{cases} 4\arctan\left(\frac{\sqrt{\widetilde{d}}}{2}\right)^2, & \sup_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| < 4, \\ 4\arctan\left(\frac{2}{\sqrt{\widetilde{d}}}\right)^2, & \inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| > 4. \end{cases} \qquad \square$$

Hence, as $\arctan(t)^2 < \frac{\pi^2}{16}$ for $t \in [0,1)$ and $t \in i\mathbb{R}$, the inequality $\sup_{x_\Sigma\in\Sigma} d(x_\Sigma) < \frac{\pi^2}{4}$ holds in both cases. Thus, if $|\widetilde{d}(x_\Sigma)| < 4$, then the assertion follows from Theorem 5.20 and $\inf_{x_\Sigma\in\Sigma}|\widetilde{d}(x_\Sigma)| > 4$, then the assertion is a consequence of Theorem 5.21.

Corollary 5.22 is particularly interesting in the case that $\widetilde{\eta}, \widetilde{\tau} \in \mathbb{R}$. Then, the conditions of Corollary 5.22 reduce to $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 \neq \pm 4$. The two excluded cases $\widetilde{d} = -4$ and $\widetilde{d} = 4$ are called the confinement case and the critical case, respectively; cf. the discussion below (1.3). We show in Corollary 7.5 that by an additional scaling of the strongly localized potentials one can also approximate Dirac operators with δ-shell potentials in the confinement case and therefore all Dirac operators with δ-shell potentials and noncritical electrostatic and Lorentz scalar interaction strengths can be approximated in the norm resolvent sense by Dirac operators with strongly localized potentials. Moreover, we provide a counterexample in Theorem 6.1 in the critical case.

6 Counterexamples

In Theorem 5.20 we showed that for $V = \eta I_N + \tau\beta$, $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$, $d = \eta^2 - \tau^2$, the condition

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < \frac{\pi^2}{4}$$

guarantees that H_{V_ε} converges in the norm resolvent sense for $\varepsilon \to 0$ to $H_{\widetilde{V}\delta_\Sigma}$ with $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, where $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$. In this chapter we show that this condition is optimal by providing counterexamples in the case of constant interaction strengths, i.e. $\eta, \tau \in \mathbb{R}$. As already mentioned in the introduction, we have to consider the case $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4$, which is the so-called critical case, and $\widetilde{d} \neq 4$, separately. From Proposition 3.15 we know that $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint if $\widetilde{d} \neq 4$. However, if $\widetilde{d} \neq 4$, then Proposition 3.15 is not applicable and $H_{\widetilde{V}\delta_\Sigma}$ is generally not self-adjoint. Thus, in the latter case it is not meaningful to ask the question whether H_{V_ε} converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$. However, under certain assumptions regarding Σ, the operator $H_{\widetilde{V}\delta_\Sigma}$ is essentially self-adjoint. The first counterexample given in Theorem 6.1 deals with this situation. In particular, we show the following: If Σ is a compact and smooth hypersurface, $d \geq \frac{\pi^2}{4}$ and $\widetilde{d} = 4$ (i.e. $d = (2k+1)^2 \frac{\pi^2}{4}$, $k \in \mathbb{N}_0$), then H_{V_ε} does not converge in the norm resolvent sense to the closure of $H_{\widetilde{V}\delta_\Sigma}$. Afterwards, we consider the case where Σ is an affine hyperplane in $\mathbb{R}^\theta$ and find out that in this situation $\sigma(H_{V_\varepsilon}) = \mathbb{R}$ if $\eta, \tau \in \mathbb{R}$ are chosen such that $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$ and $\varepsilon > 0$ is sufficiently small. Furthermore, by combining this result with known spectral properties of $H_{\widetilde{V}\delta_\Sigma}$ we show in Corollary 6.6 that if Σ is an affine hyperplane, $d \geq \frac{\pi^2}{4}$, $d \neq (2k+1)^2\pi^2$ for $k \in \mathbb{N}_0$ and $\widetilde{d} \neq 4$, then H_{V_ε} does not converge to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense. Finally, we transfer this result in Theorem 6.7 to the case where Σ is a special C^2-surface which contains a flat part. The mentioned counterexamples are particularly interesting since it is known in various situations which are included in the counterexamples that H_{V_ε} converges in the strong resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$; see [18, Theorem 7.2], [24, Theorem 2.6] and [74, Theorem 2.1].

Theorem 6.1. *Let $\Sigma \subset \mathbb{R}^\theta$ be a compact C^∞-smooth hypersurface, q be as in (4.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in \mathbb{R}$, $d = \eta^2 - \tau^2 \geq \frac{\pi^2}{4}$ such that $d \neq (2k+1)^2\pi^2$ for $k \in \mathbb{N}_0$ is fulfilled and V_ε be defined by (4.3). Moreover, let $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ with $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$ and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = 4$. Then, $H_{\widetilde{V}\delta_\Sigma}$ is not self-adjoint but essentially self-adjoint. Furthermore, H_{V_ε} does not converge in the norm resolvent sense to the closure of $H_{\widetilde{V}\delta_\Sigma}$ for $\varepsilon \to 0$.*

Proof. The claims regarding the (essential) self-adjointness follow from [13, Theorem 4.11] for $\theta = 2$ and from [19, Theorem 3.1 (ii)] for $\theta = 3$. Thus, it only remains to prove the non-convergence statement. By Proposition 2.24 we can w.l.o.g. assume $m > 0$. If Σ is compact, then $\operatorname{supp} V_\varepsilon \subset \Omega_\varepsilon$ is compact. Hence, according to [54, Theorem 3.27 (ii)] V_ε induces a compact operator from $H^1(\mathbb{R}^\theta;\mathbb{C}^N)$ to $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. Moreover, we know from Proposition 3.3 (iii) that $(H-z)^{-1}$ is bounded acting as an operator from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta;\mathbb{C}^N)$. In turn, the resolvent difference

$$(H-z)^{-1} - (H_{V_\varepsilon} - z)^{-1} = (H_{V_\varepsilon} - z)^{-1} V_\varepsilon (H-z)^{-1}$$

is compact in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$, which shows $\sigma_{\mathrm{ess}}(H_{V_\varepsilon}) = \sigma_{\mathrm{ess}}(H) = (-\infty,-m] \cup [m,\infty)$. Equipped with this preliminary observation, we now prove the claim about the non-convergence by contradiction. We assume that H_{V_ε} converges in the norm resolvent sense to $\overline{H_{\widetilde{V}\delta_\Sigma}}$ for $\varepsilon \to 0$. Then, Proposition 2.25 (ii) yields

$$\sigma_{\mathrm{ess}}(\overline{H_{\widetilde{V}\delta_\Sigma}}) = (-\infty,-m] \cup [m,\infty).$$

Furthermore, [13, Theorem 1.2 and Theorem 1.3] and [20] give us $-\frac{\widetilde{\tau}}{\widetilde{\eta}} m \in \sigma_{\mathrm{ess}}(\overline{H_{\widetilde{V}\delta_\Sigma}})$. However, in the current case $\widetilde{\eta}^2 - \widetilde{\tau}^2 = 4$ and therefore $-\frac{\widetilde{\tau}}{\widetilde{\eta}} m \in (-m,m)$ which leads to a contradiction. □

Next, we consider the case $\Sigma = \kappa(\mathbb{R}^{\theta-1} \times \{y_0\}) = \Sigma_{y_0,\kappa}$ with $\kappa \in \mathrm{SO}(\theta)$ and $y_0 \in \mathbb{R}$, i.e. Σ is an affine hyperplane in $\mathbb{R}^\theta$. Moreover, we assume $V = \eta I_N + \tau\beta$ with $\eta,\tau \in \mathbb{R}$ such that $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$. We show in Theorem 6.5 that under this set of assumptions $\sigma(H_{V_\varepsilon}) = \mathbb{R}$ for $\varepsilon > 0$ sufficiently small. In Corollary 6.6 we utilize this knowledge to show that in this situation H_{V_ε} does not converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$.

We start by applying the coordinate transformation $\widetilde{x}(x) = \kappa^T x - y_0 e_\theta$, where $x \in \mathbb{R}^\theta$ and e_θ is the θ-th Euclidean unit vector in $\mathbb{R}^\theta$, to H_{V_ε}. This transformation turns the operator H_{V_ε} into the unitarily equivalent Dirac operator

$$\begin{aligned}\widetilde{H}_{V_\varepsilon} u(\widetilde{x}) &= H_{V_\varepsilon} u(\widetilde{x}(x)) \\ &= -i\alpha \cdot \nabla_x u(\widetilde{x}(x)) + m\beta u(\widetilde{x}(x)) + V_\varepsilon(x) u(\widetilde{x}(x)) \\ &= -i\widetilde{\alpha} \cdot \nabla_{\widetilde{x}} u(\widetilde{x}) + m\beta u(\widetilde{x}) + V_\varepsilon(x) u(\widetilde{x}), \quad \widetilde{x} = \widetilde{x}(x) \in \mathbb{R}^\theta,\end{aligned}$$

for $u \in \operatorname{dom} \widetilde{H}_{V_\varepsilon} = H_{V_\varepsilon} = H^1(\mathbb{R}^\theta;\mathbb{C}^N)$, where $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots \widetilde{\alpha}_\theta)$ is defined as in (5.21), i.e. $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\theta)$ with $\widetilde{\alpha}_l = \alpha \cdot \kappa e_j$, $j \in \{1, \cdots \theta\}$, where e_j is the j-th Euclidean unit vector. Furthermore, for $\widetilde{x} = (\widetilde{x}', \widetilde{x}_\theta) = \widetilde{x}(x) \in \mathbb{R}^\theta$ we get

$$x = \kappa\widetilde{x} + y_0\kappa e_\theta = \kappa(\widetilde{x}', y_0) + \widetilde{x}_\theta \kappa e_\theta, \text{ where } \kappa(\widetilde{x}', y_0) \in \Sigma \text{ and } \kappa e_\theta \perp \Sigma.$$

In particular, combining this observation with the definition of V_ε in (4.3), and introducing $\widetilde{q}$ as the zero extension of q to $\mathbb{R}$ with $q \in L^\infty((-1,1);\mathbb{R})$ as in (4.1), we obtain

$$V_\varepsilon(x) = V\frac{\widetilde{q}\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big)}{\varepsilon} \quad \text{for } \widetilde{x} = (\widetilde{x}', \widetilde{x}_\theta) = \widetilde{x}(x) \in \mathbb{R}^\theta. \tag{6.1}$$

Hence,

$$\begin{aligned} &\operatorname{dom} \widetilde{H}_{V_\varepsilon} = H^1(\mathbb{R}^\theta;\mathbb{C}^N), \\ &\widetilde{H}_{V_\varepsilon} u(\widetilde{x}) = -i\widetilde{\alpha}\cdot\nabla_{\widetilde{x}} u(\widetilde{x}) + m\beta u(\widetilde{x}) + V\frac{\widetilde{q}\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big)}{\varepsilon} u(\widetilde{x}), \quad \widetilde{x} = (\widetilde{x}', \widetilde{x}_\theta) \in \mathbb{R}^\theta. \end{aligned} \tag{6.2}$$

Applying the Fourier transform with respect to the first $\theta - 1$ variables, which is denoted by $\mathcal{F}_1$ and defined in Section 2.1 (xvii), and identifying $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ with the Bochner space $L^2(\mathbb{R}^{\theta-1}; L^2(\mathbb{R};\mathbb{C}^N))$, see Proposition 2.18 (iii), we get similarly as in [18, eq. (2.3) and the text below] that $\mathcal{F}_1 \widetilde{H}_{V_\varepsilon} \mathcal{F}_1^{-1}$ is unitarily equivalent to the decomposable operator in $L^2(\mathbb{R}^{\theta-1}; L^2(\mathbb{R};\mathbb{C}^N))$ which is induced by the mapping

$$\mathbb{R}^{\theta-1} \ni \xi' \mapsto H_{V_\varepsilon,\xi'}$$

with

$$H_{V_\varepsilon,\xi'} := \widetilde{\alpha}'\cdot\xi' - i\widetilde{\alpha}_\theta \frac{d}{d\widetilde{x}_\theta} + m\beta + V\frac{\widetilde{q}\big(\frac{\cdot}{\varepsilon}\big)}{\varepsilon}, \quad \operatorname{dom} H_{V_\varepsilon,\xi'} := H^1(\mathbb{R};\mathbb{C}^N), \tag{6.3}$$

for $\xi' \in \mathbb{R}^{\theta-1}$; cf. Proposition 2.20. The notation $\widetilde{\alpha}'\cdot\xi'$ was introduced in (5.22). By Proposition 2.20 we can analyse the spectrum of H_{V_ε} by studying the spectrum of $H_{V_\varepsilon,\xi'}$ for $\xi' \in \mathbb{R}^{\theta-1}$. Next, we introduce the operators

$$\begin{aligned} &H_{0,\xi'} := \widetilde{\alpha}'\cdot\xi' - i\widetilde{\alpha}_\theta \frac{d}{d\widetilde{x}_\theta} + m\beta, && \operatorname{dom} H_{0,\xi'} := H^1(\mathbb{R};\mathbb{C}^N), \\ &V_R^\varepsilon : L^2(\mathbb{R};\mathbb{C}^N) \to L^2((-1,1);\mathbb{C}^N), && V_R^\varepsilon u(\widetilde{x}_\theta) := \tfrac{1}{\sqrt{\varepsilon}} u(\varepsilon\widetilde{x}_\theta), \end{aligned}$$

and

$$V_L^\varepsilon : L^2((-1,1);\mathbb{C}^N) \to L^2(\mathbb{R};\mathbb{C}^N), \qquad V_L^\varepsilon u(\widetilde{x}_\theta) := \begin{cases} \frac{1}{\sqrt{\varepsilon}} u\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big), & |\widetilde{x}_\theta| < \varepsilon, \\ 0, & |\widetilde{x}_\theta| \geq \varepsilon. \end{cases}$$

Then,

$$H_{V_\varepsilon,\xi'} = H_{0,\xi'} + V_L^\varepsilon V q V_R^\varepsilon.$$

Let us start analysing $H_{V_\varepsilon,\xi'}$ by studying the operator $H_{0,\xi'}$. Thereby, we proceed in a similar way as in [18, Section 2]. Applying the (one-dimensional) Fourier transform shows that $H_{0,\xi'}$ is unitarily equivalent to the matrix multiplication operator induced by the matrix-valued function

$$M_{\xi'} : \mathbb{R} \to \mathbb{C}^{N\times N}, \qquad M_{\xi'}(\xi_\theta) = \widetilde{\alpha}\cdot(\xi', \xi_\theta) + m\beta.$$

This implies that the operator $H_{0,\xi'}$ is self-adjoint. The spectrum of $H_{0,\xi'}$ is given by the closure of the image of the eigenvalue curves corresponding to $M_{\xi'}$; see [36, Proposition 1]. Using the rules for the Dirac matrices from (5.23) one concludes

$$(M_{\xi'}(\xi_\theta))^2 = (|\xi'|^2 + \xi_\theta^2 + m^2)I_N$$

for $\xi = (\xi', \xi_\theta) \in \mathbb{R}^\theta$. Moreover, by the structure of the Dirac matrices, $M_{\xi'}(\xi_\theta)$ cannot be a multiple of I_N if $\xi \neq 0$ or $m \neq 0$; cf. Definition 3.1 and (5.21) . Thus, the eigenvalues of $M_{\xi'}(\xi_\theta)$ are given by $\pm\sqrt{|\xi'|^2 + \xi_\theta^2 + m^2}$, and hence the spectrum of $H_{0,\xi'}$ is given by $(-\infty, -\sqrt{|\xi'|^2 + m^2}] \cup [\sqrt{|\xi'|^2 + m^2}, \infty)$. Applying the rules for the Dirac matrices again yields

$$H_{0,\xi'}^2 = \Big(-\frac{d^2}{d\widetilde{x}_\theta^2} + |\xi'|^2 + m^2 \Big) I_N, \quad \operatorname{dom} H_{0,\xi'}^2 = H^2(\mathbb{R}; \mathbb{C}^N).$$

It is well known that

$$(H_{0,\xi'}^2 - z^2)^{-1} f(t) = \int_{-\infty}^{\infty} \frac{i e^{i|t-s|\sqrt{z^2 - m^2 - |\xi'|^2}}}{2\sqrt{z^2 - m^2 - |\xi'|^2}} f(s)\, ds, \quad f \in L^2(\mathbb{R}, \mathbb{C}^N),\ t \in \mathbb{R};$$

see for instance [66, eq. (8.7)]. Note also that we switched from the variable $\widetilde{x}_\theta$ to t for convenience. The resolvent representation of $H_{0,\xi'}^2$ implies

$$\begin{aligned}(H_{0,\xi'} - z)^{-1} f(t) &= (H_{0,\xi'} + z)(H_{0,\xi'}^2 - z^2)^{-1} f(t) \\ &= \int_{-\infty}^{\infty} \Big(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \Big) \frac{i e^{i|t-s|\sqrt{z^2 - m^2 - |\xi'|^2}}}{2} f(s)\, ds\end{aligned}$$

for $f \in L^2(\mathbb{R}; \mathbb{C}^N)$ and $t \in \mathbb{R}$. Using Lemma 3.10 (i) one obtains that $z \in \rho(H_{0,\xi'})$ is in the point spectrum of $H_{V_\varepsilon,\xi'}$ if and only if

$$-1 \in \sigma_p(\mathfrak{D}_{\varepsilon,\xi'}(z) V q) \quad \text{with} \quad \mathfrak{D}_{\varepsilon,\xi'}(z) = V_R^\varepsilon (H_{0,\xi'} - z)^{-1} V_L^\varepsilon. \tag{6.4}$$

The operator $\mathfrak{D}_{\varepsilon,\xi'}(z) : L^2((-1,1); \mathbb{C}^N) \to L^2((-1,1); \mathbb{C}^N)$ has the explicit representation

$$\mathfrak{D}_{\varepsilon,\xi'}(z) g(t) = \int_{-1}^{1} \Big(\frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{\sqrt{z^2 - m^2 - |\xi'|^2}} + \widetilde{\alpha}_\theta \operatorname{sign}(t-s) \Big) \frac{i e^{i\varepsilon|t-s|\sqrt{z^2 - m^2 - |\xi'|^2}}}{2} g(s)\, ds$$

for $g \in L^2((-1,1); \mathbb{C}^N)$ and $t \in (-1,1)$. Recall that this operator was already derived in a different way in Section 5.1.1; cf. Lemma 5.1, Proposition 5.2 and Definition 5.3. In the following lemma we compare this operator with the operator $\mathfrak{H}_{0,\xi'/|\xi'|}(z)$ which was defined in (5.30) and which is for $\xi' \in \mathbb{R}^{\theta-1} \setminus \{0\}$, $g \in L^2((-1,1); \mathbb{C}^N)$ and $t \in (-1,1)$ given by

$$\mathfrak{H}_{0,\xi'/|\xi'|} g(t) = \int_{-1}^{1} \Big(\widetilde{\alpha}' \cdot \frac{\xi'}{|\xi'|} + i\widetilde{\alpha}_\theta \operatorname{sign}(t-s) \Big) \frac{1}{2} g(s)\, ds.$$

Lemma 6.2. *Let $\varepsilon > 0$, $0 \neq \xi' \in \mathbb{R}^{\theta-1}$ and $z \in (-|m|, |m|)$. Then,*

$$\begin{aligned}\|\mathfrak{H}_{0,\xi'/|\xi'|} - \mathfrak{D}_{\varepsilon,\xi'}(z)\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}&\\ \leq \sqrt{N}\Big(4\varepsilon\sqrt{m^2+|\xi'|^2} + \frac{2|m|}{|\xi'|} + \frac{m^2}{2|\xi'|^2}\Big).&\end{aligned}$$

Proof. Similarly as in Lemma 5.5, we prove the result by estimating the difference of the kernels of both operators. This difference is given by

$$\begin{aligned}\frac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|}\bigg(1 - \frac{i|\xi'|e^{i\varepsilon|t-s|\sqrt{z^2-m^2-|\xi'|^2}}}{\sqrt{z^2-m^2-|\xi'|^2}}\bigg) - \frac{i(m\beta + zI_N)e^{i\varepsilon|t-s|\sqrt{z^2-m^2-|\xi'|^2}}}{2\sqrt{z^2-m^2-|\xi'|^2}}&\\ + i\frac{\widetilde{\alpha}_\theta}{2}\operatorname{sign}(t-s)(1 - e^{i\varepsilon|t-s|\sqrt{z^2-m^2-|\xi'|^2}}).&\end{aligned} \tag{6.5}$$

Before we estimate the individual terms of this sum, we mention that for a unitary self-adjoint matrix $A \in \mathbb{C}^{N\times N}$ the Frobenius-norm is given by

$$|A|^2 = \operatorname{tr}(AA^*) = \operatorname{tr}(I_N) = N.$$

This applies in particular to $\widetilde{\alpha}'\cdot\frac{\xi'}{|\xi'|}$, $\widetilde{\alpha}_\theta$, β and I_N. Hence, the third term in (6.5) is bounded by

$$\frac{\sqrt{N}}{2}\Big|1 - e^{i\varepsilon|t-s|\sqrt{z^2-m^2-|\xi'|^2}}\Big| \leq \varepsilon\sqrt{N}\sqrt{m^2+|\xi'|^2-z^2} \leq \varepsilon\sqrt{N}\sqrt{m^2+|\xi'|^2},$$

where we used $z \in (-|m|, |m|)$ and $|t-s| \leq 2$. The second term in (6.5) can be estimated by

$$\frac{\sqrt{N}(|m|+|z|)}{2\sqrt{|\xi'|^2+m^2-z^2}} \leq \frac{\sqrt{N}|m|}{|\xi'|}.$$

Finally, we estimate the first term in (6.5) by

$$\begin{aligned}&\frac{\sqrt{N}}{2}\Big|1 - e^{i\varepsilon|t-s|\sqrt{z^2-m^2-|\xi'|^2}}\Big| + \frac{\sqrt{N}\big(\sqrt{|\xi'|^2+m^2-z^2} - |\xi'|\big)}{2\sqrt{|\xi'|^2+m^2-z^2}}\\ &\leq \varepsilon\sqrt{N}\sqrt{m^2+|\xi'|^2} + \frac{\sqrt{N}(m^2-z^2)}{2\sqrt{|\xi'|^2+m^2-z^2}(\sqrt{|\xi'|^2+m^2-z^2}+|\xi'|)}\\ &\leq \varepsilon\sqrt{N}\sqrt{m^2+|\xi'|^2} + \frac{\sqrt{N}m^2}{4|\xi'|^2},\end{aligned}$$

where we used again $z \in (-|m|, |m|)$ and $|t-s| \leq 2$. Combining these results shows that the kernel of $\mathfrak{H}_{0,\xi'/|\xi'|}(z) - \mathfrak{D}_{\varepsilon,\xi'}(z)$ can be estimated by the expression

$\sqrt{N}\big(2\varepsilon\sqrt{m^2+|\xi'|^2}+\frac{|m|}{|\xi'|}+\frac{m^2}{4|\xi'|^2}\big)$. This estimate and the Schur test, see [44, Chapter III, Example 2.4], yield

$$\begin{aligned}\|\mathfrak{H}_{0,\xi'/|\xi'|}-\mathfrak{D}_{\varepsilon,\xi'}(z)\|&_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)}\\ &\leq \int_{-1}^{1}\sqrt{N}\Big(2\varepsilon\sqrt{m^2+|\xi'|^2}+\frac{|m|}{|\xi'|}+\frac{m^2}{2|\xi'|^2}\Big)\,dt\\ &=\sqrt{N}\Big(4\varepsilon\sqrt{m^2+|\xi'|^2}+\frac{2|m|}{|\xi'|}+\frac{m^2}{2|\xi'|^2}\Big).\end{aligned}$$

. □

Lemma 6.3. *Let q be as in (4.1), $V=\eta I_N+\tau\beta$ with $\eta,\tau\in\mathbb{R}$, $d=\eta^2-\tau^2>0$, and $\xi'\in\mathbb{R}^{\theta-1}\setminus\{0\}$. Then, the (nonzero) eigenvalues of $\mathfrak{H}_{0,\xi'/|\xi'|}Vq$ are given by $\lambda_k=\frac{2\sqrt{d}}{(2k+1)\pi}$, $k\in\mathbb{Z}$.*

Proof. Let $\lambda\in\mathbb{C}\setminus\{0\}$ and $g\in L^2((-1,1);\mathbb{C}^N)$ such that $\mathfrak{H}_{0,\xi'/|\xi'|}Vqg=\lambda g$, i.e.

$$\int_{-1}^{1}\big(\widetilde{\alpha}'\cdot\tfrac{\xi'}{|\xi'|}+i\mathrm{sign}(t-s)\widetilde{\alpha}_\theta\big)\frac{1}{2}Vq(s)g(s)\,ds=\lambda g(t)\quad\forall t\in(-1,1).$$

Differentiating both sides gives us

$$i\widetilde{\alpha}_\theta Vqg=\lambda g'. \tag{6.6}$$

Hence, $g(t)=\exp\big(\frac{i\widetilde{\alpha}_\theta}{\lambda}VQ(t)\big)v$ for a $v\in\mathbb{C}^N\setminus\{0\}$, where Q is the primitive function of q chosen in such a way that $Q(1)=-Q(-1)=\frac{1}{2}$; cf. (4.37). Note that the equation $\mathfrak{H}_{0,\xi'/|\xi'|}Vqg=\lambda g$ implies

$$\begin{aligned}g(1)+g(-1)&=\int_{-1}^{1}\frac{\widetilde{\alpha}'\cdot\xi'}{|\xi'|\lambda}Vq(s)g(s)\,ds\\ &=-\frac{\widetilde{\alpha}'\cdot\xi'}{|\xi'|}i\widetilde{\alpha}_\theta\int_{-1}^{1}\frac{i\widetilde{\alpha}_\theta}{\lambda}Vq(s)\exp\big(\tfrac{i\widetilde{\alpha}_\theta}{\lambda}VQ(t)\big)v\,ds.\end{aligned}\tag{6.7}$$

Calculating the integral on the right-hand side leads to

$$g(1)+g(-1)=2\cos\big(\tfrac{\widetilde{\alpha}_\theta}{2\lambda}V\big)v=2\frac{\widetilde{\alpha}'\cdot\xi'}{|\xi'|}\widetilde{\alpha}_\theta\sin\big(\tfrac{\widetilde{\alpha}_\theta}{2\lambda}V\big)v=\frac{\widetilde{\alpha}'\cdot\xi'}{|\xi'|\lambda}V\mathrm{sinc}\big(\tfrac{\widetilde{\alpha}_\theta}{2\lambda}V\big)v.$$

Noticing $(\widetilde{\alpha}_\theta V)^2=(\widetilde{\alpha}_\theta(\eta I_N+\tau\beta))^2=dI_N$ yields

$$\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big)v=\frac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N+\tau\beta)\mathrm{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)v.$$

Thus, $v \neq 0$ implies

$$0 = \det\big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big) I_N - \tfrac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N + \tau\beta)\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)\big).$$

Now, using the rules from (5.23) and (5.24) shows that this is equivalent to

$$\begin{aligned}
0 &= \det\Big(\big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big) I_N - \tfrac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N + \tau\beta)\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)\big) \\
&\qquad\qquad \cdot \beta\big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big) I_N - \tfrac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N + \tau\beta)\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)\big)\Big) \\
&= \det\Big(\big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big) I_N - \tfrac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N + \tau\beta)\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)\big) \\
&\qquad\qquad \cdot \big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big) I_N + \tfrac{\widetilde{\alpha}'\cdot\xi'}{2|\xi'|\lambda}(\eta I_N + \tau\beta)\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)\big)\beta\Big) \\
&= \det\Big(\big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2 - \tfrac{d}{4\lambda^2}\operatorname{sinc}\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2\big)\beta\Big) \\
&= \Big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2 - \sin\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2\Big)^N \det(\beta) = \big(\cos\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2 - \sin\big(\tfrac{\sqrt{d}}{2\lambda}\big)^2\big)^N (-1)^{N/2}
\end{aligned}$$

and therefore $\lambda = \frac{2\sqrt{d}}{(2k+1)\pi} = \lambda_k$ for $k \in \mathbb{Z}$.

Finally, let us shortly argue that every λ_k, $k \in \mathbb{Z}$, is an eigenvalue of $\mathfrak{H}_{0,\xi'/|\xi'|}Vq$. Taking the the same steps as before in the reverse direction one can construct a nonzero smooth function $g \in L^2((-1,1);\mathbb{C}^N)$ which fulfils (6.6) and (6.7) for $\lambda = \lambda_k$. Integrating (6.6) over the intervals $(-1,t)$ and $(t,1)$, subtracting these two integrals and applying (6.7) gives us

$$\begin{aligned}
\int_{-1}^{t} i\widetilde{\alpha}_\theta Vq(s)g(s)\,ds - \int_{t}^{1} i\widetilde{\alpha}_\theta Vq(s)g(s)\,ds &= \lambda_k\big((g(t)-g(-1)) - (g(1)-g(t))\big) \\
&= 2\lambda_k g(t) - \int_{-1}^{1} \frac{\widetilde{\alpha}'\cdot\xi'}{|\xi'|} Vq(s)g(s)\,ds.
\end{aligned}$$

Hence, dividing by two and rearranging the terms leads to $\mathfrak{H}_{0,\xi'/|\xi'|}Vqg = \lambda_k g$; i.e. λ_k is an eigenvalue of $\mathfrak{H}_{0,\xi'/|\xi'|}Vq$. ☐

Lemma 6.4. *Let q be as in* (5.1)*, $V = \eta I_N + \tau\beta$ with $\eta, \tau \in \mathbb{R}$, $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$ and $H_{V_\varepsilon,\xi'}$, $\xi' \in \mathbb{R}^{\theta-1}$, be defined by* (6.3)*. Then, there exists an $\varepsilon_{\text{cex}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_{\text{cex}})$ and $z \in (-|m|, |m|)$ exists a $\xi'_{\varepsilon,z}$ with $z \in \sigma_p(H_{V_\varepsilon,\xi'_{\varepsilon,z}})$.*

Proof. Since $\eta^2 - \tau^2 > \frac{\pi^2}{4} > 0$, we have $\operatorname{sign}(\eta) = \operatorname{sign}(\eta+\tau) = \operatorname{sign}(\eta-\tau)$. Moreover, $q \geq 0$ a.e. by assumption (5.1). Therefore, Vq can be rewritten as $\operatorname{sign}(\eta)D^2$ with $D = \sqrt{q}\operatorname{diag}(\sqrt{|\eta+\tau|}I_{N/2}, \sqrt{|\eta-\tau|}I_{N/2})$. Hence, the eigenvalues of $\mathfrak{D}_{\varepsilon,\xi'}(z)Vq$ and $\operatorname{sign}(\eta)D\mathfrak{D}_{\varepsilon,\xi'}(z)D$, $\xi' \in \mathbb{R}^{\theta-1}$, coincide. Now, (6.4) shows that in order to verify the assertion, we have to prove that there exists an $\varepsilon_{\text{cex}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_{\text{cex}})$

and $z \in (-|m|,|m|)$ exists a $\xi'_{\varepsilon,z}$ with $1 \in \sigma_p(-\mathrm{sign}(\eta) D\mathfrak{D}_{\varepsilon,\xi'_{\varepsilon,z}}(z)D)$. Before we proceed with the proof, we introduce two useful functions. Let $\mathcal{S}(L^2((-1,1);\mathbb{C}^N))$ be the space of all self-adjoint and compact operators on $L^2((-1,1);\mathbb{C}^N)$ equipped with the operator norm topology. Note that $\mathfrak{D}_{\varepsilon,\xi'}(z)$ and $\mathfrak{H}_{0,\xi'/|\xi'|}$ are Hilbert-Schmidt operators, since their kernels are bounded, which implies

$$\mathrm{sign}(\eta) D\mathfrak{D}_{\varepsilon,\xi'}(z)D, \mathrm{sign}(\eta) D\mathfrak{H}_{0,\xi'/|\xi'|}D \in \mathcal{S}(L^2((-1,1);\mathbb{C}^N)).$$

The mapping

$$\lambda_0 : \mathcal{S}(L^2((-1,1);\mathbb{C}^N)) \to \mathbb{R}$$

which maps a compact self-adjoint operator to its biggest nonnegative element of the spectrum is continuous; see [33, Lemma 3.10]. Furthermore, we define for a fixed $\xi'_0 \in \mathbb{R}^{\theta-1} \setminus \{0\}$, and $z \in \mathbb{C} \setminus \mathbb{R}$ and $\varepsilon > 0$ the continuous function

$$\mu_{\varepsilon,z} : (0,\infty) \to \mathcal{S}(L^2((-1,1);\mathbb{C}^N)), \quad r \mapsto -\mathrm{sign}(\eta) D\mathfrak{D}_{\varepsilon,r\xi'_0}(z)D.$$

Our goal is to find $r \in (0,\infty)$ such that $\lambda_0 \circ \mu_{\varepsilon,z}(r) = 1$, as this implies that z is an eigenvalue of the operator $H_{V_\varepsilon,r\xi'_0}$. Since Proposition 2.29 shows that the spectra of $-\mathrm{sign}(\eta) D\mathfrak{H}_{0,\xi'/|\xi'|}(z)D$ and $-\mathfrak{H}_{0,\xi'/|\xi'|}(\eta I_N + \tau\beta)q$ coincide, Lemma 6.3 implies $\lambda_0(-\mathrm{sign}(\eta) D\mathfrak{H}_{0,\xi'/|\xi'|}D) = \frac{2\sqrt{d}}{\pi} > 1$. Moreover, as λ_0 is continuous, there exists a $\delta > 0$ such that for all $A \in \mathcal{S}(L^2((-1,1);\mathbb{C}^N))$ with

$$\|A + \mathrm{sign}(\eta) D\mathfrak{H}_{0,\xi'/|\xi'|}D\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} < \delta$$

also $\lambda_0(A) > 1$. According to Lemma 6.2 there exist $\varepsilon_{\mathrm{cex}} > 0$ and $r_{\mathrm{cex}} > 0$ such that for all $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$ and all $z \in (-|m|,|m|)$

$$\|\mu_{\varepsilon,z}(r_{\mathrm{cex}}) + \mathrm{sign}(\eta) D\mathfrak{H}_{0,\xi'/|\xi'|}D\|_{L^2((-1,1);\mathbb{C}^N)\to L^2((-1,1);\mathbb{C}^N)} < \delta,$$

implying $\lambda_0 \circ \mu_{\varepsilon,z}(r_{\mathrm{cex}}) > 1$ for all $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$ and all $z \in (-|m|,|m|)$. Moreover, it is easy to see that for fixed $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$ and $z \in (-|m|,|m|)$ the operator $\mu_{\varepsilon,z}(r)$ converges for $r \to \infty$ in the operator norm to zero. Therefore, also $\lambda_0 \circ \mu_{\varepsilon,z}(r)$ converges for $r \to \infty$ to zero. Let us summarize, for $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$ and $z \in (-|m|,|m|)$ the function $\lambda_0 \circ \mu_{\varepsilon,z}$ is continuous, $\lambda_0 \circ \mu_{\varepsilon,z}(r_{\mathrm{cex}}) > 1$ and $\lambda_0 \circ \mu_{\varepsilon,z}(r) \xrightarrow{r\to\infty} 0$. Hence, there exists an $r_{\varepsilon,z} \in (r_{\mathrm{cex}},\infty)$ such that $\lambda_0 \circ \mu_{\varepsilon,z}(r_{\varepsilon,z}) = 1$ and therefore $z \in \sigma_p(H_{V_\varepsilon,r_{\varepsilon,z}\xi'_0})$. Setting $\xi_{\varepsilon,z} := r_{\varepsilon,z}\xi'_0$ concludes the proof. □

Theorem 6.5. *Let* $\Sigma = \kappa(\mathbb{R}^{\theta-1} \times \{y_0\})$*, i.e.* Σ *is an affine hyperplane,* q *as in* (5.1), $V = \eta I_N + \tau\beta$ *with* $\eta,\tau \in \mathbb{R}$, $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$ *and* V_ε *be defined by* (4.3)*. Then, there exists an* $\varepsilon_{\mathrm{cex}} > 0$ *such that* $\sigma(H_{V_\varepsilon}) = \mathbb{R}$ *for all* $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$.

Proof. Let $\varepsilon_{\mathrm{cex}} > 0$ be chosen as in Lemma 6.4. We claim that in this case for all $z \in \mathbb{R}$ and $\varepsilon \in (0,\varepsilon_{\mathrm{cex}})$ exists a $\xi'_{\varepsilon,z} \in \mathbb{R}^{\theta-1}$ such that $z \in \sigma(H_{V_\varepsilon,\xi'_{\varepsilon,z}})$. If $z \in (-|m|,|m|)$, then this follows from Lemma 6.4. Now, let us assume that $z \in (-\infty, -|m|] \cup [|m|, \infty)$. The difference

$$H_{V_\varepsilon,0} - H_{0,0} = V\frac{\widetilde{q}(\frac{\cdot}{\varepsilon})}{\varepsilon}$$

is a compactly supported L^∞-function and therefore induces by [54, Theorem 3.27 (ii)] a compact operator from $H^1(\mathbb{R};\mathbb{C}^N) = \operatorname{dom} H_{0,0}$ to $L^2(\mathbb{R};\mathbb{C}^N)$. Moreover, the graph norm of $H_{0,0}$ is equivalent to the $H^1(\mathbb{R};\mathbb{C}^N)$-norm and hence V is relatively compact with respect to the operator $H_{0,0}$. Thus, [44, Chapter IV, Theorem 5.35] implies

$$(-\infty, -|m|] \cup [|m|, \infty) = \sigma(H_{0,0}) = \sigma_{\mathrm{ess}}(H_{0,0}) = \sigma_{\mathrm{ess}}(H_{V_\varepsilon,0}) \subset \sigma(H_{V_\varepsilon,0}).$$

Hence, the claim that for all $\varepsilon \in (0, \varepsilon_{\mathrm{cex}})$ and all $z \in \mathbb{R}$ exists a $\xi'_{\varepsilon,z}$ such that $z \in \sigma(H_{V_\varepsilon,\xi'_{\varepsilon,z}})$ is valid.

We finish the proof by showing that $z \in \sigma(H_{V_\varepsilon,\xi'_{\varepsilon,z}})$ for a fixed $\xi'_{\varepsilon,z}$ implies $z \in \sigma(H_{V_\varepsilon})$. The assertion follows from Proposition 2.20 and the text above (6.3) if we can show that for all $\delta > 0$ exists a $\gamma_\delta > 0$ such that $(z-\delta, z+\delta) \cap \sigma(H_{V_\varepsilon,\xi'}) \neq \emptyset$ for all $\xi' \in B(\xi'_{\varepsilon,z}, \gamma_\delta)$. We assume that our claim is not true. In this case there exists a $\delta' > 0$ and a sequence $(\xi'_n)_{n\in\mathbb{N}}$ such that $\xi'_n \overset{n\to\infty}{\longrightarrow} \xi'_{\varepsilon,z}$ and $(z-\delta', z+\delta') \cap \sigma(H_{V_\varepsilon,\xi'_n}) = \emptyset$ for all $n \in \mathbb{N}$. Note that for $w \in \mathbb{C} \setminus \mathbb{R}$ holds

$$\begin{aligned} &\big\|(H_{V_\varepsilon,\xi'_{\varepsilon,z}} - w)^{-1} - (H_{V_\varepsilon,\xi'_n} - w)^{-1}\big\|_{L^2(\mathbb{R};\mathbb{C}^N)\to L^2(\mathbb{R};\mathbb{C}^N)} \\ &\qquad = \big\|(H_{V_\varepsilon,\xi'_{\varepsilon,z}} - w)^{-1}\widetilde{\alpha}' \cdot (\xi'_n - \xi'_{\varepsilon,z})(H_{V_\varepsilon,\xi'_n} - w)^{-1}\big\|_{L^2(\mathbb{R};\mathbb{C}^N)\to L^2(\mathbb{R};\mathbb{C}^N)} \\ &\qquad \leq \frac{1}{(\operatorname{Im} w)^2}|\xi'_n - \xi'_{\varepsilon,z}| \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

i.e. H_{V_ε,ξ'_n} converges in the norm resolvent sense to $H_{V_\varepsilon,\xi'_{\varepsilon,z}}$. Moreover,

$$(z-\delta', z+\delta') \cap \sigma(H_{V_\varepsilon,\xi'_n}) = \emptyset, \quad n \in \mathbb{N},$$

and Proposition 2.25 (i) imply the contradiction $(z-\delta', z+\delta') \cap \sigma(H_{V_\varepsilon,\xi'_{\varepsilon,z}}) = \emptyset$. □

Using Theorem 6.5 leads us to the second counterexample concerning the norm resolvent convergence of H_{V_ε}.

Corollary 6.6. *Let the assumptions of Theorem 6.5 be satisfied and assume additionally $d \neq (2k+1)^2\pi^2$ for $k \in \mathbb{N}_0$. Moreover, let $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$, $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 \neq 4$ and $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$. Then, H_{V_ε} does not converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$ for $\varepsilon \to 0$.*

Proof. By Proposition 2.24 it is no restriction to assume $m > 0$. We prove this corollary by contradiction. Thus, we assume that H_{V_ε} converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$. Then, Theorem 6.5 and Proposition 2.25 (i) imply $\sigma(H_{\widetilde{V}\delta_\Sigma}) = \mathbb{R}$.

However, this is not the case according to [18, Theorem 6.2 (c)] ($\theta = 2$), where $H_{\widetilde{V}\delta_\Sigma}$ was defined and studied via direct integral methods, and [19, Theorem 4.1] ($\theta = 3$). Note that these two references only consider $\Sigma = \{0\} \times \mathbb{R}$ and $\Sigma = \mathbb{R}^2 \times \{0\}$, respectively. However, by a suitable rotation and translation one can transform $H_{\widetilde{V}\delta_\Sigma}$ to a Dirac operator with a δ-shell potential supported on $\{0\} \times \mathbb{R}$ or $\mathbb{R}^2 \times \{0\}$, respectively, and a different set of Dirac matrices; cf. [60, Proposition 4] and (6.2). The exact form of the Dirac matrices does not influence the spectrum and therefore the results from [18, 19] are also valid for the case $\Sigma = \kappa(\mathbb{R}^{\theta-1} \times \{y_0\})$. □

Having established a counterexample for the case where Σ is an affine hyperplane, we consider the case where Σ is a special C^2-surface containing a flat part in the next theorem.

Theorem 6.7. *Let $\Sigma \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1 and assume additionally that there exist $\kappa \in \mathrm{SO}(\theta)$, $y_0 \in \mathbb{R}$, $\delta > 0$ and $x_0' \in \mathbb{R}^{\theta-1}$ such that $\Sigma \supset \kappa(B(x_0', \delta) \times \{y_0\})$, i.e. Σ contains a flat part. Moreover, let q be as in (5.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in \mathbb{R}$, $d = \eta^2 - \tau^2 > \frac{\pi^2}{4}$ such that $d \neq (2k+1)^2\pi^2$ for $k \in \mathbb{N}_0$, V_ε be defined by (4.3), $(\widetilde{\eta}, \widetilde{\tau}) = \operatorname{tanc}\big(\frac{\sqrt{d}}{2}\big)(\eta, \tau)$, $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 \neq 4$ and $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$. Then, H_{V_ε} does not converge in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$.*

Proof. We prove this theorem by contradiction. Thus, we assume that H_{V_ε} converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}$ and proceed as follows: In *Step 1* we compare the two operators H_{V_ε} and $H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}$, where $V_\varepsilon^{\Sigma_{y_0,\kappa}}$ is defined as V_ε in (4.3) with Σ substituted by $\Sigma_{y_0,\kappa} = \kappa(\mathbb{R}^{\theta-1} \times \{y_0\})$. Then, in *Step 2*, we construct a resolvent formula for $H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}$ in terms of $(H_{V_\varepsilon} - z)^{-1}$. Finally, in *Step 3*, we use this formula to show that $H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}$ converges to $H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}$ for $\varepsilon \to 0$; this contradicts Corollary 6.6.

Step 1. As in (6.1) we can represent $V_\varepsilon^{\Sigma_{y_0,\kappa}}$, $\varepsilon \in (0,\infty)$, by

$$V_\varepsilon^{\Sigma_{y_0,\kappa}}(x) = V \frac{\widetilde{q}\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big)}{\varepsilon} \quad \text{for } x \in \mathbb{R}^\theta \text{ and } \widetilde{x} = (\widetilde{x}', \widetilde{x}_\theta) = \kappa^T x - y_0 e_\theta, \tag{6.8}$$

where $\widetilde{q}$ is the zero extension q to $\mathbb{R}$ and e_θ is the θ-th Euclidean unit vector. Next, we compare V_ε and $V_\varepsilon^{\Sigma_{y_0,\kappa}}$. To guarantee the well-definedness of V_ε we assume $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$ with $\varepsilon_{\mathrm{tub}} > 0$ as in Proposition 2.12. Let

$$x \in \kappa(B(x_0', \delta) \times (y_0 - \varepsilon_{\mathrm{tub}}, y_0 + \varepsilon_{\mathrm{tub}})).$$

Then, $x = \kappa(\widetilde{x}', y_0) + \widetilde{x}_\theta \kappa e_\theta$ for a vector $\widetilde{x} = (\widetilde{x}', \widetilde{x}_\theta) \in B(x_0', \delta) \times (-\varepsilon_{\mathrm{tub}}, \varepsilon_{\mathrm{tub}})$. In particular, through defining $x_\Sigma := \kappa(\widetilde{x}', y_0)$ we get $x_\Sigma \in \kappa(B(x_0', \delta) \times \{y_0\}) \subset \Sigma$ and that $\nu(x_\Sigma) = \kappa e_\theta$ is the corresponding normal vector at this point. Hence, by setting

$t := \widetilde{x}_\theta \in (-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}})$, we have $x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_{\varepsilon_{\text{tub}}}$, and by (4.3) and (6.8)

$$\begin{aligned} V_\varepsilon(x) &= \begin{cases} V\dfrac{q\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big)}{\varepsilon}, & |\widetilde{x}_\theta| < \varepsilon, \\ 0, & |\widetilde{x}_\theta| \geq \varepsilon, \end{cases} \\ &= V\frac{\widetilde{q}\big(\frac{\widetilde{x}_\theta}{\varepsilon}\big)}{\varepsilon} = V_\varepsilon^{\Sigma_{y_0,\kappa}}(x), \end{aligned}$$

implying

$$\begin{aligned} &V_\varepsilon \upharpoonright \kappa(B(x_0', \delta) \times (y_0 - \varepsilon_{\text{tub}}, y_0 + \varepsilon_{\text{tub}})) \\ &\qquad = V_\varepsilon^{\Sigma_{y_0,\kappa}} \upharpoonright \kappa(B(x_0', \delta) \times (y_0 - \varepsilon_{\text{tub}}, y_0 + \varepsilon_{\text{tub}})). \end{aligned} \tag{6.9}$$

In particular, if $u \in L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ and $\operatorname{supp} u \subset \kappa(B(x_0', \delta) \times (y_0 - \varepsilon_{\text{tub}}, y_0 + \varepsilon_{\text{tub}}))$, then

$$V_\varepsilon u = V_\varepsilon^{\Sigma_{y_0,\kappa}} u \text{ and if } u \in H^1(\mathbb{R}^\theta; \mathbb{C}^N), \text{ then also } H_{V_\varepsilon} u = H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} u.$$

Step 2. Based on *Step 1* we construct a resolvent for $H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}$ in terms of $(H_{V_\varepsilon} - z)^{-1}$ in this step. We start by constructing a partition of unity for the strip

$$\kappa\Big(\mathbb{R}^{\theta-1} \times \big(y_0 - \tfrac{\varepsilon_{\text{tub}}}{4}, y_0 + \tfrac{\varepsilon_{\text{tub}}}{4}\big)\Big).$$

To do so, let $(\phi_{n'}^a)_{n' \in \mathbb{Z}^{\theta-1}}$ with fixed $a \in (0, \max\{1, \frac{\delta}{3}\})$ be the partition of unity for $\mathbb{R}^{\theta-1}$ from Corollary A.3 (for $n = \theta - 1$). Moreover, let $(\vartheta_{n'}^a)_{n' \in \mathbb{Z}^{\theta-1}}$ be also chosen as in Corollary A.3. These sequences possess the following properties:

$$\operatorname{supp} \phi_{n'}^a \subset B(an', a), \operatorname{supp} \vartheta_{n'}^a \subset B(an', 3a) \text{ and } \phi_{n'}^a \vartheta_{n'}^a = \phi_{n'}^a \text{ for all } n' \in \mathbb{Z}^{\theta-1}.$$

Moreover, the W_∞^1-norms of $\phi_{n'}^a$ and $\vartheta_{n'}^a$ are uniformly bounded with respect to $n' \in \mathbb{Z}^{\theta-1}$. Now, we choose a function $\varrho \in C_0^\infty(\mathbb{R})$ such that $0 \leq \varrho \leq 1$, $\varrho = 1$ on $(y_0 - \frac{\varepsilon_{\text{tub}}}{4}, y_0 + \frac{\varepsilon_{\text{tub}}}{4})$ and $\operatorname{supp} \varrho \subset (y_0 - \frac{\varepsilon_{\text{tub}}}{2}, y_0 + \frac{\varepsilon_{\text{tub}}}{2})$, and define for $n' \in \mathbb{Z}^{\theta-1}$

$$\psi_{n'}^a := (\phi_{n'}^a \otimes \rho)(\kappa^T(\cdot) - y_0 e_\theta) \quad \text{as well as} \quad \varsigma_{n'}^a := \big(\vartheta_{n'}^a \otimes \varrho(\tfrac{\cdot}{2})\big)(\kappa^T(\cdot) - y_0 e_\theta).$$

Then, $(\psi_{n'}^a)_{n' \in \mathbb{Z}^{\theta-1}}$ is a partition of unity for $\kappa\big(\mathbb{R}^{\theta-1} \times (y_0 - \frac{\varepsilon_{\text{tub}}}{4}, y_0 + \frac{\varepsilon_{\text{tub}}}{4})\big)$, the inclusions

$$\begin{aligned} \operatorname{supp} \psi_{n'}^a &\subset \kappa\big(B(an', a) \times (y_0 - \tfrac{\varepsilon_{\text{tub}}}{2}, y_0 + \tfrac{\varepsilon_{\text{tub}}}{2})\big), \\ \operatorname{supp} \varsigma_{n'}^a &\subset \kappa\big(B(an', 3a) \times (y_0 - \varepsilon_{\text{tub}}, y_0 + \varepsilon_{\text{tub}})\big), \end{aligned}$$

hold and $\psi_{n'}^a \varsigma_{n'}^a = \psi_{n'}^a$ for all $n' \in \mathbb{Z}^{\theta-1}$. Now, let the unitary translation operator $T_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, which translates the argument of a function parallel to $\Sigma_{y_0,\kappa}$, be given by

$$T_{n'} : L^2(\mathbb{R}^\theta; \mathbb{C}^N) \to L^2(\mathbb{R}^\theta; \mathbb{C}^N), \quad T_{n'} u(x) = u(x + \kappa(an' - x_0', 0)).$$

As $T_{n'}$ translates the argument of a function parallel to $\Sigma_{y_0,\kappa}$, we see with the help of (6.8) that $V_\varepsilon^{\Sigma_{y_0,\kappa}}$ commutes with $T_{n'}$. Hence, also $H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}$ commutes with $T_{n'}$. Moreover, for all $n' \in \mathbb{Z}^{\theta-1}$ and $u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ the inclusions

$$\begin{aligned}\operatorname{supp} T_{n'}\psi_{n'}^a u &\subset \kappa\big(B(x_0',a)\times(y_0-\tfrac{\varepsilon_{\mathrm{tub}}}{2},y_0+\tfrac{\varepsilon_{\mathrm{tub}}}{2})\big),\\ \operatorname{supp} T_{n'}\varsigma_{n'}^a u &\subset \kappa\big(B(x_0',3a)\times(y_0-\varepsilon_{\mathrm{tub}},y_0+\varepsilon_{\mathrm{tub}})\big),\end{aligned}$$

are valid. These considerations and (6.9) show that we have for all $n' \in \mathbb{Z}^{\theta-1}$ and $u \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$

$$H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}\varsigma_{n'}^a u = T_{n'}^{-1}H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}T_{n'}\varsigma_{n'}^a u = T_{n'}^{-1}H_{V_\varepsilon}T_{n'}\varsigma_{n'}^a u. \tag{6.10}$$

We define $\psi := 1-\sum_{n'\in\mathbb{Z}}\psi_{n'}^a$. By definition $\psi = 0$ on $\kappa\big(\mathbb{R}^{\theta-1}\times(y_0-\frac{\varepsilon_{\mathrm{tub}}}{4},y_0+\frac{\varepsilon_{\mathrm{tub}}}{4})\big)$. Consequently, $H\psi u = H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}}\psi u$ for $u \in H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and $\varepsilon \in (0,\frac{\varepsilon_{\mathrm{tub}}}{4})$.

Next, we introduce for a fixed $z \in \mathbb{C}\setminus\mathbb{R}$ and for $\varepsilon \in (0,\frac{\varepsilon_{\mathrm{tub}}}{4})$ the two series

$$\begin{aligned}\mathscr{R}_{\varepsilon,1} &:= (H-z)^{-1}\psi + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\mathrm{st.}}\psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\varsigma_{n'}^a,\\ \mathscr{R}_{\varepsilon,2} &:= (H-z)^{-1}i(\alpha\cdot\nabla\psi) + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\mathrm{st.}}\psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}i(\alpha\cdot\nabla\varsigma_{n'}^a).\end{aligned}$$

We claim that both $\mathscr{R}_{\varepsilon,1}$ and $\mathscr{R}_{\varepsilon,2}$ are well-defined acting as operators in $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$. Setting $\mathcal{A}_{n'} := \psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\varsigma_{n'}^a$ and using the properties of $\psi_{n'}$ and $\vartheta_{n'}$ shows that for every $n_0' \in \mathbb{Z}^{\theta-1}$ exist at most $7^{\theta-1}$ indices $n' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n_0'}\mathcal{A}_{n'}^* \neq 0$ and $\mathcal{A}_{n_0'}^*\mathcal{A}_{n'} \neq 0$. Hence, Lemma C.2 implies that the family of operators $(\mathcal{A})_{n'\in\mathbb{Z}^{\theta-1}} = (\psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\varsigma_{n'}^a)_{n'\in\mathbb{Z}^{\theta-1}}$ is strongly summable and therefore

$$\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\mathrm{st.}}\psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\varsigma_{n'}^a$$

is well-defined and its norm is bounded by

$$\begin{aligned}C\sup_{n'\in\mathbb{Z}^{\theta-1}}&\big\|\psi_{n'}^a T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\varsigma_{n'}^a\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\ &\leq C\sup_{n'\in\mathbb{Z}^{\theta-1}}\big\|T_{n'}^{-1}(H_{V_\varepsilon}-z)^{-1}T_{n'}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\ &= C\big\|(H_{V_\varepsilon}-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}.\end{aligned}$$

Thus, $\mathscr{R}_{\varepsilon,1}$ is bounded by

$$C\Big(\big\|(H-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} + \big\|(H_{V_\varepsilon}-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\Big).$$

Similar arguments and the fact that the derivatives of $\varsigma^a_{n'}$ are uniformly bounded with respect to $n' \in \mathbb{Z}^{\theta-1}$ show that $\mathscr{R}_{\varepsilon,2}$ is well-defined and bounded by the same expression. Moreover, by choosing $\operatorname{Im} z$ big enough (independent of ε) we can guarantee $\|\mathscr{R}_{\varepsilon,2}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} < \frac{1}{2}$ and hence $I + \mathscr{R}_{\varepsilon,2}$ is boundedly invertible. Next, we show that $(I + \mathscr{R}_{\varepsilon,2})^{-1}\mathscr{R}_{\varepsilon,1}$ is the inverse of $H^{\Sigma_{y_0,\kappa}}_{V_\varepsilon} - z$. We do this by determining $(I + \mathscr{R}_{\varepsilon,2})^{-1}\mathscr{R}_{\varepsilon,1}(H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)u$ for $u \in H^1(\mathbb{R}^\theta;\mathbb{C}^N)$. The product rule gives us

$$\varsigma^a_{n'} H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} u = H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} \varsigma^a_{n'} u + i(\alpha \cdot \nabla \varsigma^a_{n'})u \quad \text{and} \quad \psi H u = H\psi u + i(\alpha \cdot \nabla\psi)u.$$

Combining this observation with (6.10) yields

$$\begin{aligned}
&(I + \mathscr{R}_{\varepsilon,2})^{-1}\mathscr{R}_{\varepsilon,1}(H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)u \\
&= (I + \mathscr{R}_{\varepsilon,2})^{-1}\Big((H - z)^{-1}\psi(H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)u \\
&\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{V_\varepsilon} - z)^{-1} T_{n'} \varsigma^a_{n'} (H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)u\Big) \\
&= (I + \mathscr{R}_{\varepsilon,2})^{-1}\Big((H - z)^{-1}(H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)\psi u \\
&\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{V_\varepsilon} - z)^{-1} T_{n'} (H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)\varsigma^a_{n'} u \\
&\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} (H - z)^{-1} i(\alpha \cdot \nabla\psi)u \\
&\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{V_\varepsilon} - z)^{-1} T_{n'} i(\alpha \cdot \nabla \varsigma^a_{n'})u\Big) \\
&= (I + \mathscr{R}_{\varepsilon,2})^{-1}\Big((H - z)^{-1}(H - z)\psi u \\
&\qquad + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{V_\varepsilon} - z)^{-1} T_{n'} T^{-1}_{n'} (H_{V_\varepsilon} - z) T_{n'} \varsigma^a_{n'} u + \mathscr{R}_{\varepsilon,2} u\Big) \\
&= (I + \mathscr{R}_{\varepsilon,2})^{-1}\Big(\psi u + \sum_{n' \in \mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} \varsigma^a_{n'} u + \mathscr{R}_{\varepsilon,2} u\Big) \\
&= (I + \mathscr{R}_{\varepsilon,2})^{-1}(I + \mathscr{R}_{\varepsilon,2})u = u.
\end{aligned} \tag{6.11}$$

Hence, $(I + \mathscr{R}_{\varepsilon,2})^{-1}\mathscr{R}_{\varepsilon,1}$ is indeed the inverse of $H^{\Sigma_{y_0,\kappa}}_{V_\varepsilon} - z$.

Step 3. Similarly as we defined $\mathscr{R}_{\varepsilon,1}$ and $\mathscr{R}_{\varepsilon,2}$, we also introduce

$$\mathscr{R}_{0,1} := (H-z)^{-1}\psi + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{\widetilde{V}\delta_\Sigma} - z)^{-1} T_{n'} \varsigma^a_{n'},$$

$$\mathscr{R}_{0,2} := (H-z)^{-1} i(\alpha\cdot\nabla\psi) + \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \psi^a_{n'} T^{-1}_{n'} (H_{\widetilde{V}\delta_\Sigma} - z)^{-1} T_{n'} i(\alpha\cdot\nabla\varsigma^a_{n'}).$$

Applying Lemma C.2 shows on the one hand that these operators are bounded by

$$C\Big(\big\|(H-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} + \sup_{n'\in\mathbb{Z}^{\theta-1}} \big\|T^{-1}_{n'}(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} T_{n'}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\Big)$$
$$= C\Big(\big\|(H-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} + \big\|(H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\Big)$$

and on the other hand that the differences $\mathscr{R}_{\varepsilon,2} - \mathscr{R}_{0,2}$ and $\mathscr{R}_{\varepsilon,1} - \mathscr{R}_{0,1}$ are bounded by

$$C\big\|\,(H_{V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \xrightarrow{\varepsilon\to 0} 0.$$

Hence, $(H_{V_\varepsilon^{\Sigma_{y_0,\kappa}}} - z)^{-1} = (I + \mathscr{R}_{\varepsilon,2})^{-1}\mathscr{R}_{\varepsilon,1}$ converges to $(I+\mathscr{R}_{0,2})^{-1}\mathscr{R}_{0,1}$ in the operator norm. It remains to prove $(I+\mathscr{R}_{0,2})^{-1}\mathscr{R}_{0,1} = (H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}} - z)^{-1}$. We start by noticing that $H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}$ also commutes with $T_{n'}$, $n'\in\mathbb{Z}^{\theta-1}$, since $T_{n'}$ translates the argument of a function only along $\Sigma_{y_0,\kappa}$ and $\widetilde{V}$ is constant. According to Lemma 5.19 $u \in \operatorname{dom} H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}$ (or equivalently $T_{n'}u = u((\cdot)+\kappa(an'-x_0',0)) \in \operatorname{dom} H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}$ for a $n'\in\mathbb{Z}^{\theta-1}$) implies $T_{n'}\varsigma^a_{n'}u = \varsigma^a_{n'}((\cdot)+\kappa(an'-x_0',0))u((\cdot)+\kappa(an'-x_0',0)) \in \operatorname{dom} H_{\widetilde{V}\delta_\Sigma}$ and

$$H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}\varsigma^a_{n'}u = T^{-1}_{n'} H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}} T_{n'}\varsigma^a_{n'}u = T^{-1}_{n'} H_{\widetilde{V}\delta_\Sigma} T_{n'}\varsigma^a_{n'}u,$$
$$\varsigma^a_{n'} H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}} u = H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}}\varsigma^a_{n'}u + i(\alpha\cdot\nabla\varsigma^a_{n'})u.$$

Using these observations one shows $(H_{\widetilde{V}\delta_{\Sigma_{y_0,\kappa}}} - z)^{-1} = (I+\mathscr{R}_{0,2})^{-1}\mathscr{R}_{0,1}$ in the same way as in (6.11). ☐

7 Consequences of the approximation results

In this chapter we present various consequences of the approximation results shown in Chapter 4 and Chapter 5. In particular, in Section 7.1, we show a scheme to approximate Dirac operators with δ-shell potentials that induce confinement; cf. [65] for the one-dimensional counterpart. Furthermore, in Section 7.2, we transfer results from Dirac operators with δ-shell potentials to Dirac operators with strongly localized potentials to study their discrete and essential spectrum.

7.1 Approximation of Dirac operators with δ-shell potentials that induce confinement

In Corollary 5.22 we saw that all Dirac operators with δ-shell potentials and interaction matrices of the form $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$, $\widetilde{\eta}, \widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$, with $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2$, satisfying

$$\sup_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| < 4 \quad \text{or} \quad \inf_{x_\Sigma \in \Sigma} |\widetilde{d}(x_\Sigma)| > 4$$

can be approximated by Dirac operators with strongly localized potentials. However, this excludes the case where $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = -4$. According to Proposition 3.15 (ii) this is a particular interesting case as it induces confinement, i.e. the operator $H_{\widetilde{V}\delta_\Sigma}$ splits into the orthogonal sum $H_{\widetilde{V}}^+ \oplus H_{\widetilde{V}}^-$ with

$$\begin{aligned} \operatorname{dom} H_{\widetilde{V}}^\pm &= \{u_\pm \in H^1(\Omega_\pm;\mathbb{C}^N) : (2I_N \mp i(\alpha\cdot\nu)\widetilde{V})\boldsymbol{t}_\Sigma^\pm u_\pm = 0\} \subset L^2(\Omega_\pm;\mathbb{C}^N), \\ H_{\widetilde{V}}^\pm u_\pm &= -i(\alpha\cdot\nabla)u_\pm + m\beta u_\pm, \end{aligned}$$

which means on a physical level that Σ becomes impermeable for a particle. This raises the question whether there is also a way to approximate such Dirac operators by Dirac operators with strongly localized potentials. Inspired by the rescaling formula in (5.6) one would have to choose $V = \eta I_N + \tau\beta$ with $d = \eta^2 - \tau^2 = -\infty$ as the interaction matrix in the approximating operators to obtain a Dirac operator with δ-shell potential and $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = -4$ in the limit. We rigorously realize this idea by choosing ε-dependent interaction strengths η_ε and τ_ε such that $\eta_\varepsilon^2 - \tau_\varepsilon^2 \xrightarrow{\varepsilon\to 0} -\infty$. The same approach was used in [65, Chapter 3] when dealing with the one-dimensional case.

We begin by explaining the setting. We choose q as in (5.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ and assume that V_ε is defined as in (4.3). Moreover, we set $d = \eta^2 - \tau^2$, assume

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < 0 \quad \text{and} \quad \operatorname{sign}(\tau) = \text{const.}, \tag{7.1}$$

and fix a continuous function $f : (0, \varepsilon_{\text{tub}}) \to (0, \infty)$, with $\varepsilon_{\text{tub}} > 0$ from Proposition 2.12, which satisfies the condition

$$\lim_{\varepsilon \to 0} f(\varepsilon) = \infty \quad \text{and} \quad \lim_{\varepsilon \to 0} f(\varepsilon)^4 \varepsilon^{1/2 - r} = 0 \tag{7.2}$$

for an $r \in (0, \frac{1}{2})$. We also introduce the ε-depend interaction strengths

$$(\eta_\varepsilon, \tau_\varepsilon) = f(\varepsilon)(\eta, \tau) \quad \text{and} \quad d_\varepsilon = \eta_\varepsilon^2 - \tau_\varepsilon^2 = f(\varepsilon)^2 d. \tag{7.3}$$

Then, $f(\varepsilon)V = \eta_\varepsilon I_N + \tau_\varepsilon \beta$. Now, we are interested in the norm resolvent convergence of the operator $H_{f(\varepsilon)V_\varepsilon}$. Note that for a fixed $\varepsilon_0 \in (0, \varepsilon_{\text{tub}})$ $H_{f(\varepsilon_0)V_\varepsilon}$ converges by Theorem 5.20 in the norm resolvent sense to $H_{\widetilde{V}_{\varepsilon_0}\delta_\Sigma}$ for $\varepsilon \to 0$ with

$$\widetilde{V}_{\varepsilon_0} := \operatorname{tanc}\big(\tfrac{\sqrt{d_{\varepsilon_0}}}{2}\big) f(\varepsilon_0) V.$$

Our goal is to prove that the limit of $H_{f(\varepsilon)V_\varepsilon}$ is the operator $H_{\widetilde{V}\delta_\Sigma}$ with

$$\begin{aligned} \widetilde{V} = \lim_{\varepsilon \to 0} \widetilde{V}_\varepsilon &= \lim_{\varepsilon \to 0} \operatorname{tanc}\big(\tfrac{\sqrt{d_\varepsilon}}{2}\big) f(\varepsilon) V \\ &= \lim_{\varepsilon \to 0} \frac{2 \tan\big(\frac{f(\varepsilon)\sqrt{d}}{2}\big)}{f(\varepsilon)\sqrt{d}} f(\varepsilon) V = \frac{2i}{\sqrt{d}} V = \frac{2}{\sqrt{|d|}} V. \end{aligned}$$

Consequently, $\widetilde{V} = \widetilde{\eta} I_N + \widetilde{\tau}\beta$ with $(\widetilde{\eta}, \widetilde{\tau}) = \frac{2}{\sqrt{|d|}}(\eta, \tau)$ and therefore $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = -4$, i.e. $H_{\widetilde{V}\delta_\Sigma}$ induces confinement.

We start by proving several preparatory statements.

Lemma 7.1. *Let q be as in (5.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ satisfy (7.1), V_ε be defined by (4.3), $f(\varepsilon)$ be the function from (7.2) and $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, let $\widetilde{V}_\varepsilon = \operatorname{tanc}(\frac{\sqrt{d_\varepsilon}}{2}) f(\varepsilon) V$, $\widetilde{V} = \frac{2}{\sqrt{|d|}} V$ and $d_{\inf} := \inf_{x_\Sigma \in \Sigma} \sqrt{|d(x_\Sigma)|} > 0$, where $d = \eta^2 - \tau^2$ and $d_\varepsilon = f(\varepsilon)^2 d$. Then, $H_{\widetilde{V}_\varepsilon \delta_\Sigma}$ and $H_{\widetilde{V}\delta_\Sigma}$ are self-adjoint and there exists an $\varepsilon_{\text{conf},1} \in (0, \varepsilon_{\text{tub}})$ such that*

$$\|(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} - (H_{\widetilde{V}_\varepsilon\delta_\Sigma} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C f(\varepsilon) e^{-f(\varepsilon) d_{\inf}}$$

for all $\varepsilon \in (0, \varepsilon_{\text{conf},1})$.

Proof. The interaction strengths corresponding to the interaction matrix $\widetilde{V}_\varepsilon$ are given by $(\widetilde{\eta}_\varepsilon, \widetilde{\tau}_\varepsilon) = \operatorname{tanc}(\frac{\sqrt{d_\varepsilon}}{2})(\eta_\varepsilon, \tau_\varepsilon)$. Using $(\eta_\varepsilon, \tau_\varepsilon) = f(\varepsilon)(\eta, \tau)$ and $d(x_\Sigma) < 0$, $x_\Sigma \in \Sigma$, yields

$$\begin{aligned}
\widetilde{d}_\varepsilon(x_\Sigma) &= \widetilde{\eta}_\varepsilon(x_\Sigma)^2 - \widetilde{\tau}_\varepsilon(x_\Sigma)^2 \\
&= 4\tan\Big(\tfrac{f(\varepsilon)\sqrt{d(x_\Sigma)}}{2}\Big)^2 \\
&= -4\tanh\Big(f(\varepsilon)\tfrac{\sqrt{|d(x_\Sigma)|}}{2}\Big)^2 \\
&\leq -4\tanh\big(f(\varepsilon)\tfrac{d_{\inf}}{2}\big)^2 \\
&< 0
\end{aligned}$$

for all $x_\Sigma \in \Sigma$. Moreover, by construction $\widetilde{d} = -4$. Thus, Proposition 3.14 implies that both $H_{\widetilde{V}_\varepsilon \delta_\Sigma}$ and $H_{\widetilde{V}\delta_\Sigma}$ are self-adjoint, and that $I + \mathcal{C}_z\widetilde{V}_\varepsilon$ as well as $I + \mathcal{C}_z\widetilde{V}$ are continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$. Moreover, the resolvent formulas

$$\begin{aligned}
(H_{\widetilde{V}_\varepsilon \delta_\Sigma} - z)^{-1} &= (H - z)^{-1} - \Phi_z \widetilde{V}_\varepsilon (I + \mathcal{C}_z \widetilde{V}_\varepsilon)^{-1}\Phi_z^*, \\
(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} &= (H - z)^{-1} - \Phi_z \widetilde{V} (I + \mathcal{C}_z \widetilde{V})^{-1}\Phi_z^*,
\end{aligned}$$

hold by Proposition 3.14. Thus, the difference of the resolvents is given by

$$\begin{aligned}
&-\Phi_z\widetilde{V}_\varepsilon(I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\Phi_z^* + \Phi_z\widetilde{V}(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_z^* \\
&\quad = \Phi_z(\widetilde{V} - \widetilde{V}_\varepsilon)(I + \mathcal{C}_z\widetilde{V})^{-1}\Phi_z^* + \Phi_z\widetilde{V}_\varepsilon\big((I + \mathcal{C}_z\widetilde{V})^{-1} - (I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\big)\Phi_z^*.
\end{aligned} \tag{7.4}$$

Simple calculations show

$$\begin{aligned}
\widetilde{V}_\varepsilon &= \operatorname{tanc}\big(\tfrac{\sqrt{d_\varepsilon}}{2}\big) f(\varepsilon)V = 2\frac{\tan\big(\frac{\sqrt{d_\varepsilon}}{2}\big)}{\sqrt{d_\varepsilon}} f(\varepsilon)V \\
&= 2\frac{\tanh\big(f(\varepsilon)\frac{\sqrt{|d|}}{2}\big)}{f(\varepsilon)\sqrt{|d|}} f(\varepsilon)V = \tanh\big(f(\varepsilon)\tfrac{\sqrt{|d|}}{2}\big)\widetilde{V}.
\end{aligned}$$

Thus, applying Proposition 2.2 (iv) and (v) yields

$$\begin{aligned}
\|\widetilde{V}_\varepsilon - \widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} &\leq C\|\tanh\big(f(\varepsilon)\tfrac{\sqrt{|d|}}{2}\big) - 1\|_{W^1_\infty(\Sigma)}\|\widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \\
&\leq C\|\tanh\big(f(\varepsilon)\tfrac{\cdot}{2}\big) - 1\|_{W^1_\infty((d_{\inf},\infty))}\|\sqrt{|d|}\|_{W^1_\infty(\Sigma)}\|\widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \\
&\leq C\|\tanh\big(f(\varepsilon)\tfrac{\cdot}{2}\big) - 1\|_{W^1_\infty((d_{\inf},\infty))}.
\end{aligned}$$

The expression $\|\tanh\big(f(\varepsilon)\frac{\cdot}{2}\big) - 1\|_{W^1_\infty((d_{\inf},\infty))}$ is smaller than

$$\|\tanh\big(f(\varepsilon)\tfrac{\cdot}{2}\big) - 1\|_{L^\infty((d_{\inf},\infty))} + \frac{f(\varepsilon)}{2}\|\tanh'\big(f(\varepsilon)\tfrac{\cdot}{2}\big)\|_{L^\infty((d_{\inf},\infty))}.$$

Next, we estimate the two summands separately. We start with the first one. For $t \in (d_{\text{inf}}, \infty)$ the inequality

$$|\tanh\big(f(\varepsilon)\tfrac{t}{2}\big) - 1| = \frac{2}{1 + e^{f(\varepsilon)t}} \leq 2e^{-f(\varepsilon)t} < 2e^{-f(\varepsilon)d_{\text{inf}}}$$

is valid. Hence, $\| \tanh\big(f(\varepsilon)\tfrac{\cdot}{2}\big) - 1\|_{L^\infty((d_{\text{inf}},\infty))} \leq 2e^{-f(\varepsilon)d_{\text{inf}}}$. For the second term we obtain

$$|\tanh'\big(f(\varepsilon)\tfrac{t}{2}\big)| = \frac{1}{\cosh(f(\varepsilon)t/2)^2} \leq e^{-f(\varepsilon)t} < e^{-f(\varepsilon)d_{\text{inf}}} \qquad \forall t \in (d_{\text{inf}}, \infty).$$

Consequently, $\| \tanh\big(f(\varepsilon)\tfrac{\cdot}{2}\big) - 1\|_{W^1_\infty((d_{\text{inf}},\infty))} \leq f(\varepsilon)e^{-f(\varepsilon)d_{\text{inf}}}$ and, in turn,

$$\|\widetilde{V}_\varepsilon - \widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \leq Cf(\varepsilon)e^{-f(\varepsilon)d_{\text{inf}}}.$$

In particular, if $\varepsilon_{\text{conf},1} \in (0, \varepsilon_{\text{tub}})$ is small enough, then Proposition 2.2 (iii) and Proposition 3.8 (i) give us for $\varepsilon \in (0, \varepsilon_{\text{conf},1})$

$$\begin{aligned}&\|(I + \mathcal{C}_z\widetilde{V})^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \\ &\qquad \cdot \|\mathcal{C}_z(\widetilde{V}_\varepsilon - \widetilde{V})\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \leq C\|\widetilde{V}_\varepsilon - \widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^N)} \leq \tfrac{1}{2}.\end{aligned}$$

Moreover, Proposition 2.28 yields for $I + \mathcal{C}_z\widetilde{V}_\varepsilon = I + \mathcal{C}_z\widetilde{V} + \mathcal{C}_z(\widetilde{V}_\varepsilon - \widetilde{V})$

$$\|(I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \leq 2\|(I + \mathcal{C}_z\widetilde{V})^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)}. \tag{7.5}$$

Using the estimates from above, Proposition 3.6 (i) and (7.4) yield

$$\begin{aligned}&\|(H_{\widetilde{V}\delta_\Sigma} - z)^{-1} - (H_{\widetilde{V}_\varepsilon} - z)^{-1}\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\ &\qquad \leq \|\Phi_z(\widetilde{V}_\varepsilon - \widetilde{V})(I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\Phi_z^*\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\ &\qquad\quad + \|\Phi_z\widetilde{V}_\varepsilon\big((I + \mathcal{C}_z\widetilde{V})^{-1} - (I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\big)\Phi_z^*\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \\ &\qquad \leq C\|\widetilde{V}_\varepsilon - \widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})} \\ &\qquad\quad + \|(I + \mathcal{C}_z\widetilde{V})^{-1} - (I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \\ &\qquad \leq Cf(\varepsilon)e^{-f(\varepsilon)d_{\text{inf}}} \\ &\qquad\quad + \|(I + \mathcal{C}_z\widetilde{V})^{-1}\mathcal{C}_z(\widetilde{V}_\varepsilon - \widetilde{V})(I + \mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \\ &\qquad \leq C\big(f(\varepsilon)e^{-f(\varepsilon)d_{\text{inf}}} + \|\widetilde{V}_\varepsilon - \widetilde{V}\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}\big) \\ &\qquad \leq Cf(\varepsilon)e^{-f(\varepsilon)d_{\text{inf}}},\end{aligned}$$

which completes the proof. □

Lemma 7.2. *Let q be as in* (5.1), $V = \eta I_N + \tau\beta$ *with* $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ *satisfy* (7.1), $f(\varepsilon)$ *be as in* (7.2), $\varepsilon_{\mathrm{conf},1}$ *be as in the previous lemma,* $z \in \mathbb{C}\setminus\mathbb{R}$, *and* $B_0(z)$ *and* $C_0(z)$ *be the operators from* (4.26) *and* (4.21). *Then,*

$$\begin{aligned}\big\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 1/2} &\leq Cf(\varepsilon)^2,\\ \big\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0} &\leq Cf(\varepsilon),\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_{\mathrm{conf},1})$.

Proof. The proof of both results is very similar. Thus, we only verify the first norm estimate. We give some remarks on how to show the second result at the end of the proof. By Proposition 3.15 (iii), Proposition 4.13 and Lemma 7.1, the operator $I + B_0(z)f(\varepsilon)Vq$ is continuously invertible in $\mathcal{B}^{1/2}(\Sigma)$ and from (4.48) we get

$$\begin{aligned}&f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z) =\\ &\qquad f(\varepsilon)Vq\cos\big((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ)\mathfrak{J}(I+\mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\Phi_{\overline{z}}^*,\end{aligned}$$

with $\widetilde{V}_\varepsilon$ from Lemma 7.1 and $Q(t) = -\frac{1}{2} + \int_{-1}^t q(t)\,dt$ for $t \in (-1,1)$; see (4.37). We know from Proposition 3.6 that $\Phi_{\overline{z}}^*$ acts as a bounded operator from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^{1/2}(\Sigma;\mathbb{C}^N)$. Moreover, from the proof of the previous lemma, see (7.5), we also know that $\|(I+\mathcal{C}_z\widetilde{V}_\varepsilon)^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)}$ is uniformly bounded with respect to $\varepsilon \in (0,\varepsilon_{\mathrm{conf},1})$. Hence, using the boundedness of $\mathfrak{J}$ acting as an operator from $H^{1/2}(\Sigma;\mathbb{C}^N)$ to $\mathcal{B}^{1/2}(\Sigma)$, see (2.10), Proposition 2.2 (iii) and Proposition 2.19 gives us

$$\begin{aligned}&\big\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 1/2}\\ &\quad\leq C\big\|f(\varepsilon)Vq\cos\big((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ)\big\|_{1/2\to 1/2}\\ &\quad= Cf(\varepsilon)\operatorname{ess\,sup}_{t\in(-1,1)}\big\|Vq(t)\cos\big((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\big)^{-1}\\ &\qquad\qquad\cdot\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ(t))\big\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)}\\ &\quad\leq Cf(\varepsilon)\operatorname{ess\,sup}_{t\in(-1,1)}\big\|\cos\big((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\big)^{-1}\\ &\qquad\qquad\cdot\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ(t))\big\|_{W^1_\infty(\Sigma;\mathbb{C}^{N\times N})}.\end{aligned}\tag{7.6}$$

Next, we fix $t \in (-1,1)$. The identity $((\alpha\cdot\nu)V)^2 = dI_N = -|d|I_N$ and $f(\varepsilon) > 0$ lead to

$$\begin{aligned}&\cos\big((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\big)^{-1}\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ(t))\\ &\qquad= \frac{\cosh(f(\varepsilon)\sqrt{|d|}Q(t)) + (\alpha\cdot\nu)\frac{V}{\sqrt{d}}\sinh(f(\varepsilon)\sqrt{|d|}Q(t))}{\cosh(f(\varepsilon)\frac{\sqrt{|d|}}{2})}.\end{aligned}\tag{7.7}$$

Applying Proposition 2.2 (v) and using $\nu \in C_b^1(\Sigma;\mathbb{R}^\theta) \subset W_\infty^1(\Sigma;\mathbb{R}^\theta)$ lets us estimate

$$\begin{aligned}
&\left\| \frac{\cosh(f(\varepsilon)\sqrt{|d|}Q(t)) + (\alpha\cdot\nu)\frac{V}{\sqrt{d}}\sinh(f(\varepsilon)\sqrt{|d|}Q(t))}{\cosh(f(\varepsilon)\frac{\sqrt{|d|}}{2})} \right\|_{W_\infty^1(\Sigma;\mathbb{C}^{N\times N})} \\
&\leq \left\| \frac{\cosh(f(\varepsilon)\sqrt{|d|}Q(t))}{\cosh(f(\varepsilon)\frac{\sqrt{|d|}}{2})} \right\|_{W_\infty^1(\Sigma;\mathbb{C}^{N\times N})} + C\left\| \frac{\sinh(f(\varepsilon)\sqrt{|d|}Q(t))}{\cosh(f(\varepsilon)\frac{\sqrt{|d|}}{2})} \right\|_{W_\infty^1(\Sigma;\mathbb{C}^{N\times N})} \\
&\leq C\big\|f(\varepsilon)\sqrt{|d|}\big\|_{W^1(\Sigma)}\bigg(\left\| \frac{\cosh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))} \\
&\qquad\qquad + \left\| \frac{\sinh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))}\bigg) \\
&\leq Cf(\varepsilon)\bigg(\left\| \frac{\cosh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))} + \left\| \frac{\sinh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))}\bigg).
\end{aligned} \tag{7.8}$$

Furthermore, $|Q(t)| \leq \frac{1}{2}$ for $t \in (-1,1)$ since $q \geq 0$ a.e. and $\int_{-1}^1 q(s)\,ds = 1$ by (5.1). Thus,

$$\left\| \frac{\cosh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))} + \left\| \frac{\sinh((\cdot)Q(t))}{\cosh(\frac{\cdot}{2})} \right\|_{W_\infty^1((0,\infty))} < \infty.$$

Hence, using (7.7) and plugging (7.8) into (7.6) yields the first inequality of the claim.

Now, we shortly comment on how to prove the second inequality. According to Proposition 3.15 (iii) and Proposition 4.13, $I + B_0(z)f(\varepsilon)Vq$ is continuously invertible in $\mathcal{B}^0(\Sigma)$. Moreover, the same formula for $f(\varepsilon)Vq(I + B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)$ is valid. Since one deals with the $\mathcal{B}^0(\Sigma)$ case now, one may estimate the norm of this expression by

$$Cf(\varepsilon)\operatorname{ess\,sup}_{t\in(-1,1)} \left\|\cos\left((\alpha\cdot\nu)\tfrac{f(\varepsilon)V}{2}\right)^{-1}\exp(-i(\alpha\cdot\nu)f(\varepsilon)VQ(t))\right\|_{L^\infty(\Sigma;\mathbb{C}^{N\times N})},$$

which, in turn, can be estimated with the help of (7.7) by $Cf(\varepsilon)$. □

Lemma 7.3. *Let q be as in (5.1), $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ satisfy (7.1), $f(\varepsilon)$ be as in (7.2), $z \in \mathbb{C}\setminus\mathbb{R}$, $B_\varepsilon(z)$ be as in (4.8) and assume $m\operatorname{sign}(\tau) > 0$. Then, there exists an $\varepsilon_{\mathrm{conf},2} \in (0, \varepsilon_{ABC})$, where $\varepsilon_{ABC} > 0$ is chosen as in (4.19), such that $(I + B_\varepsilon(z)f(\varepsilon)Vq)^{-1}$ is continuously invertible in $\mathcal{B}^0(\Sigma)$ and*

$$\|f(\varepsilon)Vq(I + B_\varepsilon(z)f(\varepsilon)Vq)^{-1}\|_{0\to 0} \leq Cf(\varepsilon)^2$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conf},2})$.

Proof. We start by mentioning $0 \in \rho(H) = \mathbb{C}\setminus\big((-\infty,-|m|] \cup [|m|,\infty)\big)$ as $m \neq 0$. Thus, the operator $\widetilde{B}_\varepsilon(0) = B_\varepsilon(0)M_\varepsilon^{-1}$, $\varepsilon \in (0, \varepsilon_{ABC})$, with M_ε from (4.20), which was introduced in (4.27), is well-defined and because of (4.36) also self-adjoint in $\mathcal{B}^0(\Sigma)$. Now, let us explain the strategy of the proof. We split the proof into three steps. In *Step 1* we show the estimate

$$\left\|(f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-1}\right\|_{0\to 0} \leq f(\varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_{\varepsilon_{ABC}}), \tag{7.9}$$

where

$$D = \sqrt{q}\operatorname{diag}(\sqrt{|\eta+\tau|}I_{N/2}, \sqrt{|\eta-\tau|}I_{N/2}).$$

In *Step 2*, we use *Step 1* and $Vq = \operatorname{sign}(\tau)D\beta D$, which is valid since $q \geq 0$ a.e and $d(x_\Sigma) = \eta(x_\Sigma)^2 - \tau(x_\Sigma)^2 < 0$ for all $x_\Sigma \in \Sigma$ by assumption, to show that the expression $\left\|f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\right\|_{0\to 0}$ is bounded by $Cf(\varepsilon)$. Lastly, in *Step 3*, we perform the change from 0 to z in order to prove (7.9).

Step 1. We begin by calculating the square

$$\begin{aligned}(f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^2 = f(\varepsilon)^{-2}I + f(\varepsilon)^{-1}&\operatorname{sign}(\tau)D(\beta\widetilde{B}_\varepsilon(0) + \widetilde{B}_\varepsilon(0)\beta)D \\ &+ (D\widetilde{B}_\varepsilon(0)D)^2.\end{aligned}$$

It is clear that $(D\widetilde{B}_\varepsilon(0)D)^2$ is a nonnegative operator. We claim that the operator $\operatorname{sign}(\tau)D(\beta\widetilde{B}_\varepsilon(0) + \widetilde{B}_\varepsilon(0)\beta)D$ is also nonnegative. Using (4.8) and (4.27) yields

$$\beta\widetilde{B}_\varepsilon(0) + \widetilde{B}_\varepsilon(0)\beta = \mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon(\beta R(0) + R(0)\beta)U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon M_\varepsilon^{-1}$$

with $R(0)$ being the inverse of the free Dirac operator H. Moreover, from Definition 3.2 one easily concludes

$$\beta R(0) + R(0)\beta = R(0)H\beta R(0) + R(0)\beta = R(0)\beta(-H + 2m\beta)R(0) + R(0)\beta = 2mR(0)^2.$$

Consequently, we get with (4.8) and (4.25)

$$\beta\widetilde{B}_\varepsilon(0) + \widetilde{B}_\varepsilon(0)\beta = 2mC_\varepsilon(0)(C_\varepsilon(0))^*.$$

Therefore, the assumption $m\operatorname{sign}(\tau) > 0$ guarantees that $\operatorname{sign}(\tau)D(\beta\widetilde{B}_\varepsilon(0) + \widetilde{B}_\varepsilon(0)\beta)D$ is a nonnegative operator in $\mathcal{B}^0(\Sigma)$. Thus, the norm of $(f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-2}$ is bounded by $f(\varepsilon)^2$. Furthermore, since $f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D$ is self-adjoint, we also have

$$\begin{aligned}&\left\|(f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-1}\right\|_{0\to 0} \\ &\qquad = \left(\left\|(f(\varepsilon)^{-1}\beta + \operatorname{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-2}\right\|_{0\to 0}\right)^{1/2} \leq f(\varepsilon)\end{aligned}$$

proving (7.9).

Step 2. Next, we choose $\varepsilon_{\mathrm{conf},2} \in (0, \varepsilon_{ABC})$ sufficiently small such that

$$\|D(\widetilde{B}_\varepsilon(0) - B_\varepsilon(0))D\|_{0\to 0} = \|DB_\varepsilon(0)(M_\varepsilon^{-1} - I)D\|_{0\to 0} \leq \frac{1}{2f(\varepsilon)}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conf},2})$. This is possible according to Lemma 4.7 and Proposition 4.10. Then, by Proposition 2.28 the operator $f(\varepsilon)^{-1}\beta + \mathrm{sign}(\tau)DB_\varepsilon(0)D$ is also continuously invertible in $\mathcal{B}^0(\Sigma)$ and the estimate

$$\begin{aligned}
&\big\|(f(\varepsilon)^{-1}\beta + \mathrm{sign}(\tau)DB_\varepsilon(0)D)^{-1}\big\|_{0\to 0} \\
&\quad\leq \frac{\big\|(f(\varepsilon)^{-1}\beta + \mathrm{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-1}\big\|_{0\to 0}}{1 - \big\|(f(\varepsilon)^{-1}\beta + \mathrm{sign}(\tau)D\widetilde{B}_\varepsilon(0)D)^{-1}\big\|_{0\to 0}\|\mathrm{sign}(\tau)D(B_\varepsilon(0) - \widetilde{B}_\varepsilon(0))D\|_{0\to 0}} \\
&\quad\leq \frac{f(\varepsilon)}{1 - \frac{f(\varepsilon)}{2f(\varepsilon)}} = 2f(\varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_{\mathrm{conf},2})
\end{aligned}$$

is valid. Therefore, $I + f(\varepsilon)\mathrm{sign}(\tau)\beta DB_\varepsilon(0)D = f(\varepsilon)\beta(f(\varepsilon)^{-1}\beta + \mathrm{sign}(\tau)\beta DB_\varepsilon(0)D)$ is also continuously invertible in $\mathcal{B}^0(\Sigma)$ and its inverse is uniformly bounded by 2. Hence, by Proposition 2.29 and $Vq = \mathrm{sign}(\tau)D\beta D$ the operator $I + B_\varepsilon(0)f(\varepsilon)Vq$ is also continuously invertible in $\mathcal{B}^0(\Sigma)$ and

$$\begin{aligned}
&\big\|f(\varepsilon)Vq(I+B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\big\|_{0\to 0} \\
&\qquad = \big\|f(\varepsilon)D(I + f(\varepsilon)\mathrm{sign}(\tau)\beta DB_\varepsilon(0)D)^{-1}\mathrm{sign}(\tau)\beta D\big\|_{0\to 0} \\
&\qquad \leq Cf(\varepsilon)
\end{aligned} \tag{7.10}$$

for all $\varepsilon \in (0, \varepsilon_{\mathrm{conf},2})$.

Step 3. In this step we perform the change from 0 to $z \in \mathbb{C} \setminus \mathbb{R}$. We start with the following observation

$$\begin{aligned}
B_\varepsilon(z) - B_\varepsilon(0) &= \mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon(R(z) - R(0))U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon \\
&= z\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon R(z)R(0)U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon = zC_\varepsilon(z)A_\varepsilon(0),
\end{aligned}$$

where we used $R(z) - R(0) = zR(z)R(0)$ and (4.8). Next, we calculate

$$\begin{aligned}
&(I + B_\varepsilon(z)f(\varepsilon)Vq)(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1} \\
&\qquad = (I + (B_\varepsilon(0) + zC_\varepsilon(z)A_\varepsilon(0))f(\varepsilon)Vq)(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1} \\
&\qquad = I + zC_\varepsilon(z)A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1} \\
&\qquad = I + z(C_\varepsilon(z) - C_0(z))A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1} \\
&\qquad\quad + zC_0(z)A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1}.
\end{aligned} \tag{7.11}$$

Now, we aim to apply Proposition 2.30 for

$$\begin{aligned}
\mathcal{A} &:= I + B_\varepsilon(z)f(\varepsilon)Vq,\\
\mathcal{A}_0 &:= I + B_0(z)f(\varepsilon)Vq,\\
\mathcal{T} &:= (I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1},\\
\mathcal{K}_1 &:= z(C_\varepsilon(z) - C_0(z))A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1},\\
\mathcal{K}_2 &:= zC_0(z)A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1}.
\end{aligned}$$

Rewriting (7.11) in terms of these newly defined operators yields $\mathcal{A}\mathcal{T} = I + \mathcal{K}_1 + \mathcal{K}_2$. Moreover, we can estimate $\mathcal{K}_1$ and $(\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2$ with Proposition 4.8, Proposition 4.9, Proposition 4.10 and (7.10) by

$$\begin{aligned}
\|\mathcal{K}_1\|_{0\to 0} &= \|z(C_\varepsilon(z) - C_0(z))A_\varepsilon(0))f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\|_{0\to 0}\\
&\leq Cf(\varepsilon)\|C_\varepsilon(z) - C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0}\|A_\varepsilon(0)\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\leq Cf(\varepsilon)\varepsilon^{1/2-r}
\end{aligned}$$

and

$$\begin{aligned}
&\|(\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\|_{0\to 0}\\
&\quad= \|z(B_0(z) - B_\varepsilon(z))f(\varepsilon)Vq(I + B_0(z)f(\varepsilon)Vq)^{-1}\\
&\qquad\qquad \cdot C_0(z)A_\varepsilon(0)f(\varepsilon)Vq(I + B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\|_{0\to 0}\\
&\quad\leq Cf(\varepsilon)\|B_\varepsilon(z) - B_0(z)\|_{1/2\to 0}\\
&\qquad\qquad \cdot \|f(\varepsilon)Vq(I + B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 1/2}\|A_\varepsilon(0)\|_{0\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\leq Cf(\varepsilon)^3\varepsilon^{1/2-r}
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_{\text{conf},2})$ as $\varepsilon_{\text{conf},2}$ is chosen such that $\varepsilon_{\text{conf},2} \in (0, \varepsilon_{ABC})$. In particular, since $f(\varepsilon)^4\varepsilon^{1/2-r} \to 0$ for $\varepsilon \to 0$, see (7.2), we have $\|\mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\|_{0\to 0} \leq \frac{1}{2}$ for all $\varepsilon \in (0, \varepsilon_{\text{conf},2})$ if $\varepsilon_{\text{conf},2} \in (0, \varepsilon_{ABC})$ is sufficiently small. In turn, Proposition 2.30 implies that $I + B_\varepsilon(z)f(\varepsilon)Vq$ has the bounded right inverse

$$\big(\mathcal{T} - \mathcal{A}_0^{-1}\mathcal{K}_2\big)\big(I + \mathcal{K}_1 + (\mathcal{A}_0 - \mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\big)^{-1},$$

which is its unique inverse according to Proposition 4.1 (i). Hence, Proposition 2.28, Proposition 4.9, Lemma 7.2 and (7.10) yield

$$\begin{aligned}
&\|f(\varepsilon)Vq(I+B_\varepsilon(z)Vq)^{-1}\|_{0\to 0} \leq \|f(\varepsilon)Vq(\mathcal{T}-\mathcal{A}_0^{-1}\mathcal{K}_2)\|_{0\to 0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \big\|\big(I+\mathcal{K}_1+(\mathcal{A}_0-\mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\big)^{-1}\big\|_{0\to 0} \\
&\quad \leq \frac{\|f(\varepsilon)Vq(\mathcal{T}-\mathcal{A}_0^{-1}\mathcal{K}_2)\|_{0\to 0}}{1-\|\mathcal{K}_1+(\mathcal{A}_0-\mathcal{A})\mathcal{A}_0^{-1}\mathcal{K}_2\|_{0\to 0}} \\
&\quad \leq 2\|f(\varepsilon)Vq(\mathcal{T}-\mathcal{A}_0^{-1}\mathcal{K}_2)\|_{0\to 0} \\
&\quad = 2\big\|f(\varepsilon)Vq(I+B_\varepsilon(0)f(\varepsilon)Vq)^{-1} \\
&\qquad - zf(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)A_\varepsilon(0)f(\varepsilon)Vq(I+B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\big\|_{0\to 0} \\
&\quad \leq C\Big(1+|z|\big\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)A_\varepsilon(0)\big\|_{0\to 0}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \|f(\varepsilon)Vq(I+B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\|_{0\to 0} \\
&\quad \leq C\Big(1+\big\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to 0}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \|f(\varepsilon)Vq(I+B_\varepsilon(0)f(\varepsilon)Vq)^{-1}\|_{0\to 0} \\
&\quad \leq C(1+f(\varepsilon))f(\varepsilon) \\
&\quad \leq Cf(\varepsilon)^2.
\end{aligned}$$

□

After providing all these preliminary results, we are ready to prove the main theorem of Section 7.1.

Theorem 7.4. *Let q be as in* (5.1), $V=\eta I_N+\tau\beta$ *with* $\eta,\tau\in C_b^1(\Sigma;\mathbb{R})$ *satisfy* (7.1), V_ε *be defined by* (4.3), $f(\varepsilon)$ *be as in* (7.2) *(with* $r\in(0,1/2)$*),* $d_{\inf}$ *be as in Lemma 7.1 and* $z\in\mathbb{C}\setminus\mathbb{R}$. *Moreover, set* $\widetilde{V}=\frac{2}{\sqrt{|d|}}V$, *where* $d=\eta^2-\tau^2$. *Then, the operator* $H_{\widetilde{V}\delta_\Sigma}$ *is self-adjoint and there exists an* $\varepsilon_{\mathrm{conf}}>0$ *and a* $C>0$ *such that*

$$\big\|(H_{f(\varepsilon)V_\varepsilon}-z)^{-1}-(H_{\widetilde{V}\delta_\Sigma}-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\big(f(\varepsilon)^4\varepsilon^{1/2-r}+f(\varepsilon)e^{-d_{\inf}f(\varepsilon)}\big)$$

for all $\varepsilon\in(0,\varepsilon_{\mathrm{conf}})$. *In particular,* $H_{f(\varepsilon)V_\varepsilon}$ *converges to* $H_{\widetilde{V}\delta_\Sigma}$ *in the norm resolvent sense as* $\varepsilon\to 0$.

Proof. Before we start, let us mention that we can w.l.o.g. assume $\operatorname{sign}(\tau)m>0$ according to Proposition 2.24. Let $\varepsilon_{\mathrm{conf}}:=\min\{\varepsilon_{\mathrm{conf},1},\varepsilon_{\mathrm{conf},2}\}$ with $\varepsilon_{\mathrm{conf},1}>0$ and $\varepsilon_{\mathrm{conf},2}>0$ from Lemma 7.1 and Lemma 7.3, respectively. According to Lemma 7.1 $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and it suffices to show

$$\big\|(H_{f(\varepsilon)V_\varepsilon}-z)^{-1}-(H_{\widetilde{V}_\varepsilon\delta_\Sigma}-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq Cf(\varepsilon)^4\varepsilon^{1/2-r},$$

where $\widetilde{V}_\varepsilon = \operatorname{tanc}(\frac{\sqrt{d_\varepsilon}}{2})f(\varepsilon)V$ with $d_\varepsilon = \eta_\varepsilon^2 - \tau_\varepsilon^2$ as in (7.3). Similarly as in the proof of Theorem 4.15, applying Proposition 4.8, Proposition 4.9, Proposition 4.10, Lemma 7.2 and Lemma 7.3 lets us estimate

$$\begin{aligned}
&\big\|(H_{f(\varepsilon)V_\varepsilon} - z)^{-1} - (H_{\widetilde{V}_\varepsilon\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\le \|A_\varepsilon(z)f(\varepsilon)Vq(I+B_\varepsilon(z)f(\varepsilon)Vq)^{-1}(C_\varepsilon(z)-C_0(z))\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+\|A_\varepsilon(z)f(\varepsilon)Vq(I+B_\varepsilon(z)f(\varepsilon)Vq)^{-1}\\
&\qquad\qquad\cdot(B_\varepsilon(z)-B_0(z))f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\qquad+\|(A_\varepsilon(z)-A_0(z))f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)}\\
&\quad\le C\varepsilon^{1/2-r}\Big(\|f(\varepsilon)Vq(I+B_\varepsilon(z)f(\varepsilon)Vq)^{-1}\|_{0\to0}\\
&\qquad+\|f(\varepsilon)Vq(I+B_\varepsilon(z)f(\varepsilon)Vq)^{-1}\|_{0\to0}\\
&\qquad\qquad\cdot\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to1/2}\\
&\qquad+\|f(\varepsilon)Vq(I+B_0(z)f(\varepsilon)Vq)^{-1}C_0(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to0}\Big)\\
&\quad\le C\varepsilon^{1/2-r}(f(\varepsilon)^2+f(\varepsilon)^4+f(\varepsilon))\\
&\quad\le C\varepsilon^{1/2-r}f(\varepsilon)^4
\end{aligned}$$

for all $\varepsilon\in(0,\varepsilon_{\mathrm{conf}})$. □

We are now in a position to answer the question posed in the beginning of the current section, namely, whether there is a way to approximate a given Dirac operator with δ-shell potential which induces confinement by Dirac operators with strongly localized potentials.

Corollary 7.5. *Let q be as in (5.1), $\widetilde{V} = \widetilde{\eta}I_N + \widetilde{\tau}\beta$ with $\widetilde{\eta},\widetilde{\tau} \in C_b^1(\Sigma;\mathbb{R})$ satisfy $\widetilde{d} = \widetilde{\eta}^2 - \widetilde{\tau}^2 = -4$ and $\operatorname{sign}(\widetilde{\tau}) = \mathrm{const.}$, $f(\varepsilon)$ be as in (7.2) (with $r\in(0,1/2)$) and $z\in\mathbb{C}\setminus\mathbb{R}$. Moreover, set $V = \upsilon\widetilde{V}$ for a $\upsilon\in C_b^1(\Sigma;\mathbb{R})$ with $\inf_{x_\Sigma\in\Sigma}\upsilon(x_\Sigma)>0$, and let V_ε be defined by (4.3) and $d_{\inf}$ be as in Lemma 7.1. Then, the operator $H_{\widetilde{V}\delta_\Sigma}$ is self-adjoint and there exists an $\varepsilon_{\mathrm{conf}}>0$ and a $C>0$ such that*

$$\big\|(H_{f(\varepsilon)V_\varepsilon}-z)^{-1}-(H_{\widetilde{V}\delta_\Sigma}-z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \le C\big(f(\varepsilon)^4\varepsilon^{1/2-r}+f(\varepsilon)e^{-d_{\inf}f(\varepsilon)}\big)$$

for all $\varepsilon\in(0,\varepsilon_{\mathrm{conf}})$. In particular, $H_{f(\varepsilon)V_\varepsilon}$ converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon\to0$.

Proof. The conditions in (7.1) are fulfilled by definition. Moreover, using $\widetilde{d}=-4$ and $d=\eta^2-\tau^2=\upsilon^2(\widetilde{\eta}^2-\widetilde{\tau}^2)=\upsilon^2\widetilde{d}$ yields

$$\frac{2}{\sqrt{|d|}}V = \frac{2}{\upsilon\sqrt{|\widetilde{d}|}}\upsilon\widetilde{V} = \widetilde{V}.$$

Therefore, the assertion follows from Theorem 7.4. □

7.2 Spectra of Dirac operators with strongly localized potentials

In this section we investigate the discrete and essential spectrum of H_{V_ε}. Since many spectral results depend on the parameter $m \in \mathbb{R}$, we write in this section $H(m)$, $H_{\widetilde{V}\delta_\Sigma}(m)$ and $H_{V_\varepsilon}(m)$ instead of H, $H_{\widetilde{V}\delta_\Sigma}$ and H_{V_ε}, see Definition 3.2, Definition 3.12 and (4.4), to emphasize the m-dependence of these operators.

We start by presenting results for the case that Σ is bounded. The resolvent difference of $H_{V_\varepsilon}(m) = H(m) + V_\varepsilon$ and $H(m)$ is for $z \in \rho(H(m)) \cap \rho(H_{V_\varepsilon}(m))$ given by

$$(H(m) - z)^{-1} - (H_{V_\varepsilon}(m) - z)^{-1} = (H_{V_\varepsilon}(m) - z)^{-1} V_\varepsilon (H(m) - z)^{-1}.$$

In the case that Σ is bounded, Ω_ε, see Definition 2.7, is also bounded. Thus, V_ε is compactly supported and, in turn, induces a compact operator from $H^1(\mathbb{R}^\theta; \mathbb{C}^N)$ to $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$; cf. [54, Theorem 3.27 (ii)]. Moreover, $(H(m) - z)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^\theta; \mathbb{C}^N)$ to $H^1(\mathbb{R}^\theta; \mathbb{C}^N)$; see Proposition 3.3 (iii). Thus, also the resolvent difference is compact for $z \in \rho(H(m)) \cap \rho(H_{V_\varepsilon}(m))$. Therefore,

$$\sigma_{\mathrm{ess}}(H_{V_\varepsilon}(m)) = \sigma_{\mathrm{ess}}(H) = \sigma(H) = (-\infty, -|m|] \cup [|m|, \infty)$$

and $\sigma_{\mathrm{disc}}(H_{V_\varepsilon}(m)) \subset (-|m|, |m|)$. Next, we focus on special cases which allow us to make more precise statements concerning the discrete spectrum. Here, we are particularly interested in cases which guarantee the existence of discrete eigenvalues.

Proposition 7.6. *Let $\theta = 3$, $\Sigma \subset \mathbb{R}^3$ be the boundary of a bounded C^∞-smooth domain, $m \in \mathbb{R}$, $V = \tau I_4$ with $\tau \in \mathbb{R}$ and $H_{V_\varepsilon}(m)$ be as in the beginning of Section 7.2. Then, the following holds:*

(i) *If $\tau m \geq 0$ and $\delta \in (0, |m|)$, then $\sigma_{\mathrm{disc}}(H_{V_\varepsilon}(m)) \cap [-|m| + \delta, |m| - \delta] = \emptyset$ for $\varepsilon > 0$ sufficiently small.*

(ii) *If $\tau \neq 0$ and $M \in \mathbb{N}$, then there exists for sufficiently large $-\mathrm{sign}(\tau)m > 0$ an $\varepsilon_m > 0$ such that for all $\varepsilon \in (0, \varepsilon_m)$ the operator $H_{V_\varepsilon}(m)$ has at least M discrete eigenvalues counted with multiplicities.*

Proof. If $V = \tau I_4$, then $d = -\tau^2 \leq 0$ and therefore it follows from Theorem 5.20 that $H_{V_\varepsilon}(m)$ converges for $\varepsilon \to 0$ to $H_{\widetilde{V}\delta_\Sigma}(m)$ in the norm resolvent sense, where $\widetilde{V} = \widetilde{\tau}\beta$ with

$$\widetilde{\tau} = \operatorname{tanc}\big(\tfrac{\sqrt{d}}{2}\big)\tau = \operatorname{tanc}\big(\tfrac{\sqrt{-\tau^2}}{2}\big)\tau = \frac{2\tanh\big(\frac{|\tau|}{2}\big)}{|\tau|}\tau = 2\tanh\big(\tfrac{\tau}{2}\big).$$

Having established the convergence of $H_{V_\varepsilon}(m)$ we are able to prove (i) and (ii). We start with (i). If $\tau m \geq 0$, then also $\widetilde{\tau} m \geq 0$ and hence according to [37, Proposition 3.6 b)] the discrete spectrum of $H_{\widetilde{V}\delta_\Sigma}(m)$ is empty. Moreover, since

Σ is bounded we have $\sigma_{\text{ess}}(H_{V_\varepsilon}(m)) = (-\infty, -|m|] \cup [|m|, \infty)$ and thus by Proposition 2.25 (ii) $\sigma(H_{\widetilde{V}\delta_\Sigma}(m)) = \sigma_{\text{ess}}(H_{\widetilde{V}\delta_\Sigma}(m)) = (-\infty, -|m|] \cup [|m|, \infty)$. Consequently, Proposition 2.27 (i) implies

$$\sigma_{\text{disc}}(H_{V_\varepsilon}(m)) \cap [-|m| + \delta, |m| - \delta] = \sigma(H_{V_\varepsilon}(m)) \cap [-|m| + \delta, |m| - \delta] = \emptyset$$

for $\varepsilon > 0$ sufficiently small.
Next, we consider (ii). We obtain from [37, Theorem 2.3 (e) and Corollary 4.4] that $H_{\widetilde{V}\delta_\Sigma}(m)$ has at least M discrete eigenvalues (counted with multiplicities) for sufficiently large $-\text{sign}(\tau)m = -\text{sign}(\widetilde{\tau})m > 0$. Now, applying Proposition 2.26 yields assertion (ii). $\square$

After considering purely Lorentz scalar interaction strengths in Proposition 7.6, we consider purely electrostatic interaction strengths in the next statement.

Proposition 7.7. *Let $\theta = 3$, $\Sigma \subset \mathbb{R}^3$ be the unit sphere, $m = 1$, $V = \eta I_4$ with $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, there holds the following:*

(i) *If $|\eta| < 2\arctan(\sqrt{5} - 2)$ and $\delta \in (0, 1)$, then $\sigma_{\text{disc}}(H_{V_\varepsilon}(1)) \cap [-1 + \delta, 1 - \delta] = \emptyset$ for $\varepsilon > 0$ sufficiently small.*

(ii) *If $|\eta| > 2\arctan(\sqrt{5} - 2)$, then $\sigma_{\text{disc}}(H_{V_\varepsilon}(1)) \neq \emptyset$ for $\varepsilon > 0$ sufficiently small.*

Proof. The assumption $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ implies $d = \eta^2 < \frac{\pi^2}{4}$ and therefore Theorem 5.20 guarantees that $H_{V_\varepsilon}(1)$ converges for $\varepsilon \to 0$ to $H_{\widetilde{V}\delta_\Sigma}(1)$ in the norm resolvent sense, where $\widetilde{V} = \widetilde{\eta} I_4$ with

$$\widetilde{\eta} = \operatorname{tanc}\Big(\frac{\sqrt{d}}{2}\Big)\eta = \operatorname{tanc}\Big(\frac{\sqrt{\eta^2}}{2}\Big)\eta = \frac{2\tan\big(\frac{|\eta|}{2}\big)}{|\eta|}\eta = 2\tan\big(\tfrac{\eta}{2}\big).$$

If $|\eta| < 2\arctan(\sqrt{5} - 2)$, then $|\widetilde{\eta}| < 2\sqrt{5} - 4$ and hence [6, Theorem 1.1 and Lemma 5.2 (iii)] show that the discrete spectrum of $H_{\widetilde{V}\delta_\Sigma}(1)$ is empty. Now, the same arguments as in the proof of Proposition 7.6 (i) yield assertion (i). If the inequality $|\eta| > 2\arctan(\sqrt{5} - 2)$ holds, then $|\widetilde{\eta}| > 2\sqrt{5} - 4$ and thus $\sigma_{\text{disc}}(H_{\widetilde{V}\delta_\Sigma}(1)) \neq \emptyset$ by [5, (4.35)–(4.36) and the text below]. Consequently, Proposition 2.26 implies (ii). $\square$

Next, we consider the case where Σ is unbounded. In particular, we assume that Σ is a non-self-intersecting C^∞-smooth curve in $\mathbb{R}^2$ which coincides outside of a compact set with a broken line with opening angle 2ω, $\omega \in (0, \frac{\pi}{2})$, which is given by

$$\widetilde{\Gamma}_\omega = \{r(\cos(\omega), \sin(\omega)) : r > 0\} \cup \{r(\cos(\omega), -\sin(\omega)) : r > 0\}.$$

In this setting the discrete eigenvalues of $H_{\widetilde{V}\delta_\Sigma}(m)$ were studied in [10, Section 2.4]. In the upcoming two statements we use Theorem 5.20 to transfer these results from $H_{\widetilde{V}\delta_\Sigma}(m)$ to $H_{V_\varepsilon}(m)$. This allows us to provide conditions under which $H_{V_\varepsilon}(m)$ admits discrete eigenvalues.

Proposition 7.8. *Let $\theta = 2$ and Σ be a non-self-intersecting C^∞-curve which coincides outside of a compact set with $\widetilde{\Gamma}_\omega$ for a $\omega \in (0, \frac{\pi}{2})$, $a < 0$, $\eta = \tau = \frac{a}{m}$ and $V = V(m) = \eta I_2 + \tau\beta$ for $m > 0$. Then, the following holds:*

(i) *If $\delta \in (0, m\frac{m^2-a^2}{m^2+a^2})$, then $\sigma_{\mathrm{ess}}(H_{V_\varepsilon}(m)) \cap [-m+\delta, m\frac{m^2-a^2}{m^2+a^2} - \delta] = \emptyset$ for sufficiently small $\varepsilon > 0$.*

(ii) *If $m > 0$ is sufficiently large, then there exists an $\varepsilon_m > 0$ such that for $\varepsilon \in (0, \varepsilon_m)$ the discrete spectrum of $H_{V_\varepsilon}(m)$ is not empty.*

Proof. Under the assumptions of this proposition we have $d = \eta^2 - \tau^2 = \frac{a^2}{m^2} - \frac{a^2}{m^2} = 0$ and therefore $(\widetilde{\eta}, \widetilde{\tau}) = \mathrm{tanc}(\frac{\sqrt{d}}{2})(\eta, \tau) = (\eta, \tau)$. Thus, Theorem 5.20 implies that $H_{V_\varepsilon}(m)$ converges for $\varepsilon \to 0$ to $H_{V\delta_\Sigma}(m)$ in the norm resolvent sense. Moreover, [10, Theorem 2.3 (ii) (c)] and rescaling the results from [10, Theorem 2.7] yields

$$\sigma_{\mathrm{ess}}(H_{V\delta_\Sigma}(m)) = (-\infty, m] \cup [m\tfrac{m^2-a^2}{m^2+a^2}, \infty)$$

and $\sigma_{\mathrm{disc}}(H_{V\delta_\Sigma}(m)) \neq \emptyset$ for sufficiently large $m > 0$, respectively. Combining these results with Proposition 2.27 and Proposition 2.26 concludes the proof of (i) and (ii), respectively. □

Proposition 7.9. *Let $\theta = 2$ and Σ be a non-self-intersecting C^∞-curve which coincides outside of a compact set with $\widetilde{\Gamma}_\omega$ for a $\omega \in (0, \frac{\pi}{2})$, $\tau < 0$, $V = \tau\beta$ and $\widetilde{\tau} = 2\tanh(\tau/2)$. Moreover, assume that $\tau m < 0$. Then, the following holds:*

(i) *If $\delta \in (0, |m|\frac{4-\widetilde{\tau}^2}{4+\widetilde{\tau}^2})$, then $\sigma_{\mathrm{ess}}(H_{V_\varepsilon}(m)) \cap [-|m|\frac{4-\widetilde{\tau}^2}{4+\widetilde{\tau}^2} + \delta, |m|\frac{4-\widetilde{\tau}^2}{4+\widetilde{\tau}^2} - \delta] = \emptyset$ for sufficiently small $\varepsilon > 0$.*

(ii) *For every $M \in \mathbb{N}$ exists an $\omega_M \in (0, \frac{\pi}{2})$ and an $\varepsilon_M > 0$ such that for $\varepsilon \in (0, \varepsilon_M)$ the operator $H_{V_\varepsilon}(m)$ has at least M discrete eigenvalues with multiplicities taken into account.*

Proof. If $V = \tau I_4$, then it follows from Theorem 5.20 that $H_{V_\varepsilon}(m)$ converges for $\varepsilon \to 0$ to $H_{\widetilde{V}\delta_\Sigma}(m)$ in the norm resolvent sense, where $\widetilde{V} = \widetilde{\tau}\beta$ with $\widetilde{\tau} = 2\tanh\big(\frac{\tau}{2}\big) \in (-2, 2)$. This also shows that the assumption $\tau m < 0$ implies $\widetilde{\tau} m < 0$. Hence, according to [10, Corollary 2.5] the essential spectrum of $H_{\widetilde{V}\delta_\Sigma}(m)$ is given by

$$\sigma_{\mathrm{ess}}(H_{V_\varepsilon}(m)) = (-\infty, -|m|\tfrac{4-\widetilde{\tau}^2}{4+\widetilde{\tau}^2}] \cup [|m|\tfrac{4-\widetilde{\tau}^2}{4+\widetilde{\tau}^2}, \infty)$$

and therefore Proposition 2.27 (ii) implies assertion (i). By [10, Theorem 2.8] and [37, Theorem 2.3 (e)] there exists an $\omega_M \in (0, \frac{\pi}{2})$ such that $H_{\widetilde{V}\delta_\Sigma}(m)$ has at least M discrete eigenvalues. Thus, assertion (ii) follows from Proposition 2.26 □

We conclude this section by considering the case where the interaction matrix has the form $f(\varepsilon)V$ with $V = \eta I_N + \tau\beta$, $\eta, \tau \in \mathbb{R}$, such that $d = \eta^2 - \tau^2 < 0$ and f is the monotonically decreasing function from (7.2). Recall that in this case the conditions of Theorem 7.4 are met and therefore the operator $H_{f(\varepsilon)V_\varepsilon}(m)$ converges in the norm resolvent sense to $H_{\widetilde{V}\delta_\Sigma}(m)$ with $\widetilde{V} = \frac{2}{\sqrt{|d|}}V$, which is a Dirac operator that induces confinement; cf. Section 7.1.

Proposition 7.10. *Let $\theta = 3$, $\Sigma \subset \mathbb{R}^3$ be the boundary of a bounded C^∞-smooth domain, $V = \eta I_4 + \tau\beta$ with $\eta, \tau \in \mathbb{R}$ fulfil $d = \eta^2 - \tau^2 < 0$ and $m \neq 0$. Then, the following holds:*

(i) *If $m\tau > 0$ and $\delta \in (0, |m|)$, then $\sigma_{\mathrm{disc}}(H_{f(\varepsilon)V_\varepsilon}(m)) \cap [-|m| + \delta, |m| - \delta] = \emptyset$ for sufficiently small $\varepsilon > 0$.*

(ii) *If $\eta = 0$ and $M \in \mathbb{N}$, then for sufficiently large $-\mathrm{sign}(\tau)m > 0$ exists an $\varepsilon_m > 0$ such that for $\varepsilon \in (0, \varepsilon_m)$ the operator $H_{f(\varepsilon)V_\varepsilon}(m)$ has at least M discrete eigenvalues counted with multiplicities.*

Proof. According to Theorem 7.4 $H_{f(\varepsilon)V_\varepsilon}(m)$ converges for $\varepsilon \to 0$ to $H_{\widetilde{V}\delta_\Sigma}(m)$ in the norm resolvent sense with $\widetilde{V} = \widetilde{\eta} I_4 + \widetilde{\tau}\beta$, where $(\widetilde{\eta}, \widetilde{\tau}) = \frac{2}{\sqrt{|d|}}(\eta, \tau)$. Hence, if $\tau > 0$, then $\widetilde{\tau} > 0$ and therefore [11, Proposition 2.2] implies that the discrete spectrum of $H_{\widetilde{V}\delta_\Sigma}$ is empty if $m > 0$. Thus, for $m > 0$ assertion (i) follows from Proposition 2.27; cf. the proof of Proposition 7.6 (i) for a analogous more detailed proof. It remains to consider the case $m < 0$. The anticommutation rules from (3.1) imply $H_{f(\varepsilon)V_\varepsilon}(m) = -\beta H_{-f(\varepsilon)V_\varepsilon}(-m)\beta$ and thus this case can be reduced to the case $m > 0$. Now, let us consider (ii). Under the set of assumptions in (ii), [3, Corollary 1.15] and [37, Remark 2.1, Theorem 2.3 (e)] imply that $H_{\widetilde{V}\delta_\Sigma}(m)$ has at least M discrete eigenvalues for sufficiently large $-\mathrm{sign}(\tau)m$. Consequently, (ii) follows from Proposition 2.26. $\square$

8 Convergence of Dirac operators with semilocal potentials

In this chapter we approximate Dirac operators with δ-shell potentials by Dirac operators with so-called semilocal potentials. Before we define these semilocal potentials, we motivate their definition by results regarding one-dimensional Dirac operators.

Recall that δ_0 denotes the δ-potential supported in the point $\{0\}$. Furthermore, let $h \in L^\infty(\mathbb{R};\mathbb{R}) \cap L^1(\mathbb{R};\mathbb{R})$ with $\int_\mathbb{R} h(x)\,dx = 1$ and $h_\varepsilon := \frac{1}{\varepsilon}h(\frac{\cdot}{\varepsilon})$ for $\varepsilon > 0$. Then, the multiplication operator induced by h_ε and the projection operator defined by $u \mapsto h_\varepsilon(u, h_\varepsilon)_{L^2(\mathbb{R})}$ converge to δ_0 viewed as operators from $C_0^\infty(\mathbb{R})$ to $(C_0^\infty(\mathbb{R}))'$, i.e.

$$\begin{aligned}\lim_{\varepsilon\to 0}\, {}_{(C_0^\infty(\mathbb{R}))'}\langle h_\varepsilon u, v\rangle_{C_0^\infty(\mathbb{R})} &= {}_{(C_0^\infty(\mathbb{R}))'}\langle \delta_0 u, v\rangle_{C_0^\infty(\mathbb{R})}\\ &= u(0)v(0)\\ &= \lim_{\varepsilon\to 0}\, {}_{(C_0^\infty(\mathbb{R}))'}\langle h_\varepsilon, u\rangle_{C_0^\infty(\mathbb{R})}\, {}_{(C_0^\infty(\mathbb{R}))'}\langle h_\varepsilon, v\rangle_{C_0^\infty(\mathbb{R})}\\ &= \lim_{\varepsilon\to 0}\, {}_{(C_0^\infty(\mathbb{R}))'}\langle h_\varepsilon (u, h_\varepsilon)_{L^2(\mathbb{R})}, v\rangle_{C_0^\infty(\mathbb{R})}\end{aligned}$$

for all $u, v \in C_0^\infty(\mathbb{R})$. Similarly as in the multidimensional case, $H + Vh_\varepsilon$ converges to $H_{\widetilde{V}\delta_\Sigma}$ in the norm (or strong) resolvent sense for $V = V^* \in \mathbb{C}^{2\times 2}$; see [40, 41, 42, 67, 72] and Section 1.2. Surprisingly, it was shown in [34, Section 4] and [67] that $H + Vh_\varepsilon(\cdot, h_\varepsilon)_{L^2(\mathbb{R})}$ converges to $H_{V\delta_\Sigma}$ in the norm resolvent sense, i.e. for these kinds of nonlocal potentials there is no rescaling necessary. In [35] this idea was picked up and used as an inspiration for the two and three-dimensional setting. In the mentioned paper they considered (generalizations of) Dirac operators of the type $H + q_\varepsilon V(\cdot, q_\varepsilon)_{L^2(\mathbb{R}^\theta)}$, where $V = V^* \in \mathbb{C}^{N\times N}$ and

$$q_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon}q\big(\frac{t}{\varepsilon}\big), & x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_\varepsilon,\\ 0, & x \notin \Omega_\varepsilon,\end{cases}$$

with q as in (4.1). It turns out that this family of operators converges to the operator which is formally given by $H + V\delta_\Sigma\, {}_{C_0^\infty(\mathbb{R}^\theta)}\langle \delta_\Sigma, \cdot\rangle_{C_0^\infty(\mathbb{R}^\theta)}$ and can be realized as an unperturbed Dirac operator on $\Omega_+ \cup \Omega_-$ and via nonlocal transmission conditions on Σ, i.e. transmission conditions which involve the integral over Σ; see also [33]. Hence, on the one hand no renormalization of V is necessary but on the other hand the limit operator is not the operator $H_{V\delta_\Sigma}$, which is the operator we aim to approximate. This leads us to the definition of so-called semilocal potentials. They behave locally with respect to the surface Σ and nonlocally with respect to the normal direction of Σ and allow us to approximate $H_{V\delta_\Sigma}$ without any rescaling.

Let us recall our general setting from the beginning of Chapter 4. Let $\Sigma = \partial\Omega_\pm \subset \mathbb{R}^\theta$ be a special C^2-surface as in Definition 2.1, Ω_ε and ι be as in Definition 2.7, and $\varepsilon_{\mathrm{tub}} \in (0,\infty)$ be chosen as in Proposition 2.12. Moreover, recall from (4.1) and (4.2) that $q \in L^\infty((-1,1);\mathbb{R})$ with $\int_{-1}^1 q(t)\,dt = 1$ and $V = V^* \in W^1_\infty(\Sigma;\mathbb{C}^N)$. In this setting we define for $\varepsilon \in (0,\varepsilon_{\mathrm{tub}})$

$$
\begin{aligned}
&\underline{V_\varepsilon} : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N),\\
&\underline{V_\varepsilon}u(x) := \begin{cases} \frac{1}{\varepsilon}q\big(\frac{t}{\varepsilon}\big)V(x_\Sigma)\int_{-1}^1 u(\iota(x_\Sigma,\varepsilon s))q(s) & \\ \qquad\qquad\qquad \cdot\det(I - s\varepsilon W(x_\Sigma))\,ds, & x = \iota(x_\Sigma,t) \in \Omega_\varepsilon,\\ 0, & x \notin \Omega_\varepsilon. \end{cases}
\end{aligned}
\tag{8.1}
$$

Note that the expression $\det(I - s\varepsilon W(x_\Sigma))$, which is defined below Definition 2.11 and stems from the usage of the tubular coordinates (x_Σ,t), plays a secondary role as it converges to one for $\varepsilon \to 0$; see Proposition 2.12. However, the term is necessary to guarantee the self-adjointness of $\underline{V_\varepsilon}$.

8.1 General interactions

In this section we consider $H_{\underline{V_\varepsilon}} = H + \underline{V_\varepsilon}$ for general $V = V^* \in W^1_\infty(\Sigma;\mathbb{C}^{N\times N})$. It turns out that we can represent the resolvent of $H_{\underline{V_\varepsilon}}$ in a similar way as the resolvent of H_{V_ε} in Proposition 4.1. This allows us to transfer the convergence results from the local to the semilocal case.

We use the operators $\mathfrak{J}$, $\mathcal{I}_\varepsilon$, $\mathcal{S}_\varepsilon$, U_ε and M_ε, see (2.10), (4.5)–(4.7) and (4.20), to express $\underline{V_\varepsilon}u$ for $u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and $x = \iota(x_\Sigma,t) \in \Omega_\varepsilon$ by

$$
\begin{aligned}
\underline{V_\varepsilon}u(x) &= \frac{1}{\varepsilon}q\big(\tfrac{t}{\varepsilon}\big)V(x_\Sigma)\int_{-1}^1 u(\iota(x_\Sigma,\varepsilon s))q(s)\det(I - s\varepsilon W(x_\Sigma))\,ds\\
&= \frac{1}{\varepsilon}q\big(\tfrac{t}{\varepsilon}\big)V(x_\Sigma)\int_{-1}^1 q(s)\big(M_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon u\big)(s)(x_\Sigma)\,ds\\
&= \frac{1}{\varepsilon}q\big(\tfrac{t}{\varepsilon}\big)\big(V\mathfrak{J}^*qM_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon u\big)(x_\Sigma)\\
&= \frac{1}{\varepsilon}\big(q\mathfrak{J}V\mathfrak{J}^*qM_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon u\big)\big(\tfrac{t}{\varepsilon}\big)(x_\Sigma)\\
&= \frac{1}{\sqrt{\varepsilon}}\big(\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon q\mathfrak{J}V\mathfrak{J}^*qM_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon u\big)(x)\\
&= \big(U_\varepsilon^*\mathcal{I}_\varepsilon\mathcal{S}_\varepsilon q\mathfrak{J}V\mathfrak{J}^*qM_\varepsilon\mathcal{S}_\varepsilon^{-1}\mathcal{I}_\varepsilon^{-1}U_\varepsilon u\big)(x).
\end{aligned}
$$

For $u \in L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ and $x \notin \Omega_\varepsilon$ we trivially have

$$\underline{V_\varepsilon} u(x) = 0 = \big(U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon q \mathfrak{J} V \mathfrak{J}^* q M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon u\big)(x).$$

Thus,

$$\underline{V_\varepsilon} = U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon q \mathfrak{J} V \mathfrak{J}^* q M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon.$$

This representation implies that $\underline{V_\varepsilon}$ is a bounded operator in $L^2(\mathbb{R}^\theta,\mathbb{C}^N)$ and with the help of $(\mathcal{I}_\varepsilon \mathcal{S}_\varepsilon)^* = M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1}$, see the lines above (4.25), one gets that $\underline{V_\varepsilon}$ is self-adjoint. Hence, similarly as in Section 4.1, we define for $z \in \rho(H)$, $R(z) = (H-z)^{-1}$ and $\varepsilon \in (0, \varepsilon_{\text{tub}})$

$$\begin{aligned}
\underline{A_\varepsilon}(z) &:= R(z) U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon q \mathfrak{J} : L^2(\Sigma;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N),\\
\underline{B_\varepsilon}(z) &:= \mathfrak{J}^* q M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(z) U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon q \mathfrak{J} : L^2(\Sigma;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N),\\
\underline{C_\varepsilon}(z) &:= \mathfrak{J}^* q M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon R(z) : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N).
\end{aligned}$$

By definition, the identities

$$\underline{A_\varepsilon}(z) = A_\varepsilon(z) q \mathfrak{J}, \quad \underline{B_\varepsilon}(z) = \mathfrak{J}^* q M_\varepsilon B_\varepsilon(z) q \mathfrak{J}, \quad \underline{C_\varepsilon}(z) = \mathfrak{J}^* q M_\varepsilon C_\varepsilon(z),$$

are valid, where $A_\varepsilon(z)$, $B_\varepsilon(z)$ and $C_\varepsilon(z)$ are the operators from (4.8). Applying Proposition 3.11 for $P_L = U_\varepsilon^* \mathcal{I}_\varepsilon \mathcal{S}_\varepsilon q \mathfrak{J} V$, $P_R = \mathfrak{J}^* q M_\varepsilon \mathcal{S}_\varepsilon^{-1} \mathcal{I}_\varepsilon^{-1} U_\varepsilon$ and $\underline{V_\varepsilon} = P_L P_R$ yields the following proposition.

Proposition 8.1. *Let q and V be as in (4.1) and (4.2), $\underline{V_\varepsilon}$ be defined by (8.1) for $\varepsilon \in (0, \varepsilon_{\text{tub}})$, $z \in \rho(H)$ and $R(z) = (H-z)^{-1}$, where H is the free Dirac operator introduced in Definition 3.2. Then, $H_{\underline{V_\varepsilon}}$ is self-adjoint and the following holds:*

(i) $z \in \sigma_p(H_{\underline{V_\varepsilon}}) \iff -1 \in \sigma_p(\underline{B_\varepsilon(z)} V)$.

(ii) *If $-1 \in \rho(V \underline{B_\varepsilon(z)})$, then $z \in \rho(H_{\underline{V_\varepsilon}})$ and*

$$(H_{\underline{V_\varepsilon}} - z)^{-1} = R(z) - \underline{A_\varepsilon}(z) V (I + \underline{B_\varepsilon}(z) V)^{-1} \underline{C_\varepsilon}(z).$$

Proposition 8.1 shows that the resolvent of $H_{\underline{V_\varepsilon}}$ has a similar structure as the resolvent of H_{V_ε}; cf. Proposition 4.1 (ii). Moreover, the operators $\underline{A_\varepsilon}(z)$, $\underline{B_\varepsilon}(z)$ and $\underline{C_\varepsilon}(z)$ are strongly connected to the operators $A_\varepsilon(z)$, $B_\varepsilon(z)$ and $C_\varepsilon(z)$, respectively. We use this connection to transfer the convergence results from the local operators to the semilocal operators. Before we do so, we introduce the limit operators

$$\begin{aligned}
\underline{A_0}(z) &:= \Phi_z : L^2(\Sigma;\mathbb{C}^N) \to L^2(\mathbb{R}^\theta;\mathbb{C}^N),\\
\underline{B_0}(z) &:= \mathcal{C}_z : L^2(\Sigma;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N),\\
\underline{C_0}(z) &:= \Phi_{\overline{z}}^* : L^2(\mathbb{R}^\theta;\mathbb{C}^N) \to L^2(\Sigma;\mathbb{C}^N).
\end{aligned}$$

Proposition 8.2. *Let $z \in \rho(H)$ and $\varepsilon \in (0, \varepsilon_{ABC})$ with ε_{ABC} given by (4.19). Then, the families of operators $(\underline{A_\varepsilon}(z))_{\varepsilon\in(0,\varepsilon_{ABC})}$, $(\underline{B_\varepsilon}(z))_{\varepsilon\in(0,\varepsilon_{ABC})}$ and $(\underline{C_\varepsilon}(z))_{\varepsilon\in(0,\varepsilon_{ABC})}$ are uniformly bounded and for $r \in (0, 1/2)$ holds*

$$\begin{aligned}
\|\underline{A_\varepsilon}(z) - \underline{A_0}(z)\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\Sigma;\mathbb{C}^N)} &\le C\varepsilon^{1/2-r},\\
\|\underline{B_\varepsilon}(z) - \underline{B_0}(z)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)\to L^2(\Sigma;\mathbb{C}^N)} &\le C\varepsilon^{1/2-r},\\
\|\underline{C_\varepsilon}(z) - \underline{C_0}(z)\|_{L^2(\Sigma;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} &\le C\varepsilon^{1/2-r}.
\end{aligned}$$

Furthermore, the operators $\underline{C_\varepsilon}(z)$, $\varepsilon \in (0, \varepsilon_{ABC})$, are also well-defined acting as operators from $L^2(\mathbb{R}^\theta;\mathbb{C}^N)$ to $H^{1/2}(\Sigma;\mathbb{C}^N)$ and the corresponding operator norms are uniformly bounded.

Proof. The assertions are direct consequences of Lemma 4.7, Proposition 4.8, Proposition 4.9 and Proposition 4.10 if

$$\underline{A_0}(z) = A_0(z)q\mathfrak{J}, \quad \underline{B_0}(z) = \mathfrak{J}^*qB_0(z)q\mathfrak{J}, \quad \underline{C_0}(z) = \mathfrak{J}^*qC_0(z). \tag{8.2}$$

According to (4.24) we have $A_0(z) = \Phi_z\mathfrak{J}^*$. Hence, $A_0(z)q\mathfrak{J} = \Phi_z\mathfrak{J}^*q\mathfrak{J}$. Furthermore, $\int_{-1}^1 q(t)\,dt = 1$, (2.10) and (2.11) give us for $\psi \in L^2(\Sigma;\mathbb{C}^N)$

$$\mathfrak{J}^*q\mathfrak{J}\psi = \int_{-1}^1 q(t)(\mathfrak{J}\psi)(t)\,dt = \int_{-1}^1 q(t)\,dt\,\psi = \psi, \tag{8.3}$$

which shows $A_0(z)q\mathfrak{J} = \Phi_z = \underline{A_0}(z)$. Moreover, (4.21) and (8.3) also imply

$$\mathfrak{J}^*qC_0(z) = \mathfrak{J}^*q\mathfrak{J}\Phi_{\overline{z}}^* = \Phi_{\overline{z}}^* = \underline{C_0}(z).$$

The representation for $B_0(z)$ in (4.38) and (8.3) yield

$$\mathfrak{J}^*qB_0(z)q\mathfrak{J} = \mathfrak{J}^*qT(\alpha\cdot\nu)q\mathfrak{J} + \mathfrak{J}^*q\mathfrak{J}\mathcal{C}_z\mathfrak{J}^*q\mathfrak{J} = \mathfrak{J}^*qT(\alpha\cdot\nu)q\mathfrak{J} + \mathcal{C}_z.$$

Thus, it remains to show $\mathfrak{J}^*qT(\alpha\cdot\nu)q\mathfrak{J} = 0$. This follows from

$$\begin{aligned}
\mathfrak{J}^*qT(\alpha\cdot\nu)q\mathfrak{J}\psi &= \int_{-1}^1 q(t)\frac{i}{2}\int_{-1}^1 \operatorname{sign}(t-s)(\alpha\cdot\nu)q(s)\psi\,ds\,dt\\
&= \int_{-1}^1\int_{-1}^1 \operatorname{sign}(t-s)q(t)q(s)\,ds\,dt\frac{i}{2}(\alpha\cdot\nu)\psi = 0 \quad \forall\psi \in L^2(\Sigma;\mathbb{C}^N).
\end{aligned}$$

□

Having stated these preliminary results, which were essentially consequences of Section 4.3, we are able to present the main theorem of this section, which states conditions under which $H_{\underline{V_\varepsilon}}$ converges to $H_{V\delta_\Sigma}$ in the norm resolvent sense.

Theorem 8.3. *Let q and V be as in (4.1) and (4.2), $\varepsilon_{ABC} > 0$ be as in (4.19) and assume that for some $z \in \rho(H)$ the following conditions are fulfilled:*

(i) *There exists an $\underline{\varepsilon_{\text{conv}}} \in (0, \varepsilon_{ABC}]$ such that $(I+\underline{B_\varepsilon}(z)Vq)^{-1}$ exists for $\varepsilon \in (0, \underline{\varepsilon_{\text{conv}}})$ and is uniformly bounded in $L^2(\Sigma; \mathbb{C}^N)$.*

(ii) *The operator $I + \mathcal{C}_z V$ is bijective in $H^{1/2}(\Sigma; \mathbb{C}^N)$.*

Then, the operator $H_{V\delta_\Sigma}$ is self-adjoint, $z \in \rho(H_{V\delta_\Sigma}) \cap \rho(H_{\underline{V_\varepsilon}})$ for all $\varepsilon \in (0, \underline{\varepsilon_{\text{conv}}})$ and for any $r \in (0, \frac{1}{2})$ exists a $C > 0$ such that

$$\left\|(H_{\underline{V_\varepsilon}} - z)^{-1} - (H_{V\delta_\Sigma} - z)^{-1}\right\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for $\varepsilon \in (0, \underline{\varepsilon_{\text{conv}}})$. In particular, $H_{\underline{V_\varepsilon}}$ converges to $H_{V\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Proof. Since $I+\mathcal{C}_z V$ is continuously invertible in $H^{1/2}(\Sigma; \mathbb{C}^N)$, it follows from Proposition 3.14 that $H_{V\delta_\Sigma} - z$ is invertible and

$$\begin{aligned}(H_{V\delta_\Sigma} - z)^{-1} &= R(z) - \Phi_z V(I + \mathcal{C}_z V)^{-1}\Phi_{\overline{z}}^* \\ &= R(z) - \underline{A_0}(z)V(I + \underline{B_0}(z)V)^{-1}\underline{C_0}(z).\end{aligned}$$

The rest of the proof can be shown in exactly the same way as Theorem 4.15 by applying Proposition 8.1, Proposition 8.2 and using the spaces $H^r(\Sigma; \mathbb{C}^N)$ instead the spaces $\mathcal{B}^r(\Sigma)$. □

8.2 An explicit condition for electrostatic and Lorentz scalar interactions

Similarly as in Chapter 5, we find an explicit convergence condition for the norm resolvent convergence of $H_{\underline{V_\varepsilon}}$ if $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ in this section. Our goal is to show that the conditions (5.1) and

$$\sup_{x_\Sigma \in \Sigma} d(x_\Sigma) < 4, \quad d = \eta^2 - \tau^2, \tag{8.4}$$

guarantees norm resolvent convergence. If (8.4) holds, then Proposition 3.15 (iii) shows that (ii) of Theorem 8.3 is fulfilled for $z \in \mathbb{C} \setminus \mathbb{R}$. Thus, we only have to consider (i) of Theorem 8.3. We proceed as follows: We show that if Σ is a rotated C_b^2-graph, then the operator $(I + V\underline{B_\varepsilon}(z))^{-1}$ exists and is uniformly bounded in $L^2(\Sigma; \mathbb{C}^N)$ with respect to ε. Then, (ii) of Theorem 8.3 is fulfilled and therefore $H_{\underline{V_\varepsilon}}$ converges in the norm resolvent sense to $H_{V\delta_\Sigma}$ for $\varepsilon \to 0$. If Σ is a special C^2-surface as in Definition 2.1 one uses again a partition of unity to prove the main result of this section which is Theorem 8.9.

Recall from Section 5.1.2 that if Σ is a a rotated C_b^2-graph, then there exists a function $\zeta \in C_b^2(\mathbb{R}^\theta;\mathbb{R})$ and a rotation matrix $\kappa \in \mathrm{SO}(\mathbb{R}^\theta)$ such that

$$\Sigma = \Sigma_{\zeta,\kappa} = \{\kappa(x',\zeta(x')) : x' \in \mathbb{R}^{\theta-1}\}.$$

As in Section 8.1, we make extensive use of the results known from the case of local potentials; cf. Section 5.1. Thereby, we use the following notation: If O is one of the operators introduced in (5.11) or (5.48), which act as bounded operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$, then we define the semilocal version of O through

$$\underline{O} = \mathfrak{J}^* q O q \mathfrak{J} : L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N). \tag{8.5}$$

If O also acts as a bounded operator from $\mathcal{B}^r(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^{r'}(\mathbb{R}^{\theta-1})$ with $r, r' \in [0,1]$, then the properties of $\mathfrak{J}$ and q immediately imply

$$\|\underline{O}\|_{H^r(\Sigma;\mathbb{C}^N)\to H^{r'}(\Sigma;\mathbb{C}^N)} \leq 2\|q\|^2_{L^\infty((-1,1))}\|O\|_{r\to r'}. \tag{8.6}$$

We aim to prove that $(I + \underline{B_\varepsilon}(z)V)^{-1}$ is uniformly bounded in $L^2(\Sigma;\mathbb{C}^N)$ if ε is sufficiently small. We show this in the same way as we proved the uniform boundedness of $(I + B_\varepsilon(z)Vq)^{-1}$ in Section 5.1. As almost all the steps are identical, with the sole exception of Proposition 8.5, we keep this section as short as possible and refer to Section 5.1 for details. To shorten notation and to emphasize the connection to Section 5.1 we also write $\|\cdot\|_{\underline{r}\to\underline{r'}}$ instead of $\|\cdot\|_{H^r(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^{r'}(\mathbb{R}^{\theta-1};\mathbb{C}^N)}$ in the current section.

Proposition 8.4. *Let $x_0' \in \mathbb{R}^{\theta-1}$, ζ and κ be as in (5.42), $\zeta_{x_0'}$ be as in (5.46), q be as in (4.1), $\eta,\tau \in C_b^1(\Sigma;\mathbb{R})$, $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (5.18), $z \in \rho(H)$, $\psi \in C_b^1(\mathbb{R}^{\theta-1})$, $\varepsilon \in (0,\varepsilon_{\mathrm{gr},1})$ with $\varepsilon_{\mathrm{gr},1}$ chosen as in Lemma 5.12 and $a_\varepsilon = \varepsilon^{1/6}$. Then, the operators $\underline{E_\varepsilon}(z)$ and $[\underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z),\psi]$ act as bounded operators from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and*

$$\|\underline{E_\varepsilon}(z)\|_{\underline{0}\to\underline{1}} \leq C\frac{1+|\log(\varepsilon)|}{a_\varepsilon},$$
$$\big\|[\underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z),\psi]\big\|_{\underline{0}\to\underline{1}} \leq C\|\psi\|_{W^1_\infty(\mathbb{R}^{\theta-1})}(1+|\log(\varepsilon)|),$$
$$\big\|\chi_{B(x_0',3a_\varepsilon)}\big(\underline{D_\varepsilon^{\zeta,\kappa}}(z)Q_{\eta,\tau}^{\zeta,\kappa} - \underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\big)\chi_{B(x_0',3a_\varepsilon)}\big\|_{\underline{0}\to\underline{0}} \leq Ca_\varepsilon(1+|\log(\varepsilon)|),$$

where $C > 0$ does not depend on x_0' and ε.

Proof. The statement follows from Proposition 5.14 and (8.6). □

In Proposition 8.5 we show that $(I + \underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0'))^{-1}$ is uniformly bounded with respect to ε and $x_0' \in \mathbb{R}^{\theta-1}$. This is the only proposition in the current section where we apply a different proof strategy than in the analogous local statement given by Proposition 5.15 in Section 5.1.

Proposition 8.5. *Let $x_0' \in \mathbb{R}^{\theta-1}$, ζ and κ be as in (5.42), $\zeta_{x_0'}$ be as in (5.46), q be as in (5.1), $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$, $d = \eta^2 - \tau^2$ satisfy (8.4), $Q_{\eta,\tau}^{\zeta,\kappa}$ be as in (5.18) and $z \in \mathbb{C}\setminus\mathbb{R}$. Then, there exists an $\underline{\varepsilon_{\mathrm{gr},1}} > 0$ such that the operators $(I + \underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z)Q_{\eta,\tau}^{\zeta,\kappa}(x_0'))^{-1}$ are uniformly bounded with respect to $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{gr},1}})$ and $x_0' \in \mathbb{R}^{\theta-1}$.*

Proof. Let us first assume that $\zeta_{x_0'} \equiv y_0 \in \mathbb{R}$. Then, we get from (8.5), the comments below (5.20) and Proposition 5.2 for $\xi' \in \mathbb{R}^{\theta-1}$

$$(\mathcal{F}\underline{D_\varepsilon^{y_0,\kappa}}(z)\mathcal{F}^{-1}f)(\xi') = (\mathcal{F}\underline{D_\varepsilon^{0,\kappa}}(z)\mathcal{F}^{-1}f)(\xi') = \frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{2\sqrt{z^2 - m^2 - |\xi'|^2}}\omega(\xi', \varepsilon),$$

where $\omega(\xi', \varepsilon) = i\int_{-1}^1 \int_{-1}^1 q(t)q(s)e^{|\varepsilon(t-s)|i\sqrt{z^2-m^2-|\xi'|^2}}\,ds\,dt$ and $\widetilde{\alpha}' \cdot \xi'$ is defined as in (5.22). Hence, the operator

$$I + \mathcal{F}\underline{D_\varepsilon^{\zeta_{x_0'},\kappa}}(z)\mathcal{F}^{-1}Q_{\eta,\tau}^{\zeta,\kappa}(x_0')$$

is a matrix multiplication operator induced by the matrix-valued function

$$\mathbb{R}^{\theta-1} \ni \xi' \mapsto I_N + \frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{2\sqrt{z^2 - m^2 - |\xi'|^2}}\omega(\xi', \varepsilon)Q_{\eta,\tau}^{\zeta,\kappa}(x_0').$$

We define the function

$$\begin{aligned}&\mathbb{R}^2 \times \mathbb{R}^{\theta-1} \times (0,\infty) \ni (\widehat{\eta}, \widehat{\tau}, \xi', \varepsilon)\\ &\qquad \mapsto p(\widehat{\eta}, \widehat{\tau}, \xi', \varepsilon) := 1 + (\widehat{\eta}^2 - \widehat{\tau}^2)\frac{\omega(\xi', \varepsilon)^2}{4} + (\widehat{\eta}z + \widehat{\tau}m)\tfrac{\omega(\xi',\varepsilon)}{\sqrt{z^2-m^2-|\xi'|^2}}.\end{aligned}$$

Now, using the rules from (5.23) and $Q_{\eta,\tau}^{\zeta,\kappa}(x_0') = \eta(\varkappa_{\zeta,\kappa}(x_0'))I_N + \tau(\varkappa_{\zeta,\kappa}(x_0'))\beta$ yields

$$\begin{aligned}&\Big(I_N + \frac{\widetilde{\alpha}' \cdot \xi' + m\beta + zI_N}{2\sqrt{z^2 - m^2 - |\xi'|^2}}\omega(\xi', \varepsilon)\big(\eta(\varkappa_{\zeta,\kappa}(x_0'))I_N + \tau(\varkappa_{\zeta,\kappa}(x_0'))\beta\big)\Big)\\ &\qquad \cdot \Big(I_N - \big(\eta(\varkappa_{\zeta,\kappa}(x_0'))I_N - \tau(\varkappa_{\zeta,\kappa}(x_0'))\beta\big)\frac{\widetilde{\alpha}' \cdot \xi' + m\beta - zI_N}{2\sqrt{z^2 - m^2 - |\xi'|^2}}\omega(\xi', \varepsilon)\Big)\\ &\qquad = p(\eta(\varkappa_{\zeta,\kappa}(x_0')), \tau(\varkappa_{\zeta,\kappa}(x_0')), \xi', \varepsilon)I_N.\end{aligned}$$

Hence, if $p(\eta(\varkappa_{\zeta,\kappa}(x_0')), \tau(\varkappa_{\zeta,\kappa}(x_0')), \xi', \varepsilon) \neq 0$, then $I_N + \frac{\widetilde{\alpha}'\cdot\xi'+m\beta-zI_N}{2\sqrt{z^2-m^2-|\xi'|^2}}\omega(\xi', \varepsilon)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')$ is invertible and by applying (5.24), $\eta, \tau \in C_b^2(\Sigma;\mathbb{R})$ and $\zeta \in C_b^2(\mathbb{R}^{\theta-1};\mathbb{R})$ we are able to estimate

$$\Big|\Big(I_N + \frac{\widetilde{\alpha}' \cdot \xi' + m\beta - zI_N}{2\sqrt{z^2 - m^2 - |\xi'|^2}}\omega(\xi', \varepsilon)Q_{\eta,\tau}^{\zeta,\kappa}(x_0')\Big)^{-1}\Big| \leq \frac{C}{|p(\eta(\varkappa_{\zeta,\kappa}(x_0')), \tau(\varkappa_{\zeta,\kappa}(x_0')), \xi', \varepsilon)|},$$

where $C > 0$ does not depend on ξ', x_0', κ and ε. Moreover, there exists an $\underline{\varepsilon_{\mathrm{gr},1}} > 0$ such that

$$p_{\inf} = \inf_{(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)},\xi'\in\mathbb{R}^{\theta-1},\varepsilon\in(0,\underline{\varepsilon_{\mathrm{gr},1}})} |p(\widehat{\eta},\widehat{\tau},\xi',\varepsilon)| > 0. \tag{8.7}$$

We postpone the proof of (8.7) to the end of the proof . This estimate and Proposition 2.19 imply that for $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{gr},1}})$ the operator $I + \underline{D_\varepsilon^{y_0,\kappa}} Q_{\eta,\tau}^{\zeta,\kappa}(x_0')$ is invertible and the norm of its inverse is bounded by $\frac{C}{p_{\inf}}$, which does not depend on κ, x_0', y_0 and ε. Thus, the assertion is true if $\zeta_{x_0'} \equiv y_0 \in \mathbb{R}$. If this is not the case, then as $\Sigma_{\zeta_{x_0'},\kappa}$ is an affine hyperplane in $\mathbb{R}^\theta$, there exists an $\widetilde{y}_0(x_0') \in \mathbb{R}$ and a $\widetilde{\kappa}(x_0') \in \mathrm{SO}(\theta)$ such that $\Sigma_{\zeta_{x_0'},\kappa} = \Sigma_{\widetilde{y}_0(x_0'),\widetilde{\kappa}(x_0')}$; cf (5.57). Hence, one gets in the same way as in the proof of Proposition 5.15

$$\|(I + \underline{D_\varepsilon^{\zeta_{x_0'},\kappa}} Q_{\eta,\tau}^{\zeta,\kappa}(x_0'))^{-1}\|_{\underline{0}\to\underline{0}} = \|(I + \underline{D_\varepsilon^{\widetilde{y}_0(x_0'),\widetilde{\kappa}(x_0')}} Q_{\eta,\tau}^{\zeta,\kappa}(x_0'))^{-1}\|_{\underline{0}\to\underline{0}} \leq \frac{C}{p_{\inf}}.$$

It remains to prove (8.7). We start by introducing

$$d_{\max} := \max_{(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)}} \widehat{\eta}^2 - \widehat{\tau}^2 \quad \text{and} \quad \widetilde{\omega}(\xi',\varepsilon) = i\int_{-1}^1\int_{-1}^1 q(s)q(t)e^{-\varepsilon|t-s||\xi'|}\,ds\,dt.$$

Then, (8.4) implies $d_{\max} < 4$. Moreover, $|\widetilde{\omega}(\xi',\varepsilon)| \leq 1$ and $\widetilde{\omega}(\xi',\varepsilon)^2 < 0$ since $q \geq 0$ a.e. on $(-1,1)$ by (5.1). Thus,

$$\left|1 + (\widehat{\eta}^2 - \widehat{\tau}^2)\frac{\widetilde{\omega}(\xi',\varepsilon)^2}{4}\right| \geq \min\Big\{1 - \frac{d_{\max}}{4}, 1\Big\} > 0 \tag{8.8}$$

for all $\xi' \in \mathbb{R}^{\theta-1}$, $\varepsilon > 0$ and $(\widehat{\eta},\widehat{\tau}) \in \overline{\operatorname{ran}(\eta,\tau)}$. Furthermore, (5.32), (5.33) and $q \in L^\infty((-1,1))$ yield

$$|\omega(\xi',\varepsilon) - \widetilde{\omega}(\xi',\varepsilon)| \leq \frac{C}{1+|\xi'|}$$

for all $\xi' \in \mathbb{R}^{\theta-1}$ and $\varepsilon > 0$. Thus, since $\overline{\operatorname{ran}(\eta,\tau)}$ is bounded $\underline{\text{and } |\omega(\xi'},\varepsilon)| \leq 1$, there exists an $R > 0$ such that for all $|\xi'| \geq R$, $\varepsilon > 0$ and $(\widehat{\eta},\widehat{\tau}) \in \overline{\operatorname{ran}(\eta,\tau)}$

$$\left|(\widehat{\eta}^2 - \widehat{\tau}^2)\frac{\omega(\xi',\varepsilon)^2 - \widetilde{\omega}(\xi',\varepsilon)^2}{4} + (\widehat{\eta}z + \widehat{\tau}m)\frac{\omega(\xi',\varepsilon)}{\sqrt{z^2 - m^2 - |\xi'|^2}}\right| \leq \frac{\min\Big\{1 - \frac{d_{\max}}{4}, 1\Big\}}{2}.$$

This and (8.8) imply for all $|\xi'| \geq R$, $\varepsilon > 0$ and $(\widehat{\eta},\widehat{\tau}) \in \overline{\operatorname{ran}(\eta,\tau)}$

$$\begin{aligned}
|p(\widehat{\eta},\widehat{\tau},\xi',\varepsilon)| \geq{}& \left|1 + (\widehat{\eta}^2 - \widehat{\tau}^2)\frac{\widetilde{\omega}(\xi',\varepsilon)^2}{4}\right| \\
&- \left|(\widehat{\eta}^2 - \widehat{\tau}^2)\frac{\omega(\xi',\varepsilon)^2 - \widetilde{\omega}(\xi',\varepsilon)^2}{4} + (\widehat{\eta}z + \widehat{\tau}m)\frac{\omega(\xi',\varepsilon)}{\sqrt{z^2 - m^2 - |\xi'|^2}}\right| \\
>{}& \frac{\min\Big\{1 - \frac{d_{\max}}{4}, 1\Big\}}{2};
\end{aligned}$$

i.e.

$$\inf_{(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)},|\xi'|\geq R,\varepsilon>0}|p(\widehat{\eta},\widehat{\tau},\xi',\varepsilon)|>\frac{\min\left\{1-\frac{d_{\max}}{4},1\right\}}{2}>0. \tag{8.9}$$

Thus, it remains to consider the case $|\xi'|\leq R$. We use $\int_{-1}^{1}q(t)\,dt=1$ to get

$$|\omega(\xi',\varepsilon)-i|=\left|\int_{-1}^{1}\int_{-1}^{1}q(t)q(s)\left(e^{|\varepsilon(t-s)|i\sqrt{z^2-m^2-|\xi'|^2}}-1\right)ds\,dt\right|\leq C\varepsilon R\quad\forall|\xi'|\leq R,$$

where $C>0$ does only depend on q, z and m. Hence,

$$\left|p(\widehat{\eta},\widehat{\tau},\xi',\varepsilon)-\left(1-\frac{\widehat{\eta}^2-\widehat{\tau}^2}{4}+\frac{i(\widehat{\eta}z+\widehat{\tau}m)}{\sqrt{z^2-m^2-|\xi'|^2}}\right)\right|\leq C\varepsilon R \tag{8.10}$$

for all $|\xi'|\leq R$, $\varepsilon>0$ and $(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)}$. Next, we show

$$\inf_{(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)},|\xi'|\leq R}\left|1-\frac{\widehat{\eta}^2-\widehat{\tau}^2}{4}+\frac{i(\widehat{\eta}z+\widehat{\tau}m)}{\sqrt{z^2-m^2-|\xi'|^2}}\right|>0. \tag{8.11}$$

Since $\overline{B(0,R)}\times\overline{\operatorname{ran}(\eta,\tau)}$ is compact, it suffices to show

$$1-\frac{\widehat{\eta}^2-\widehat{\tau}^2}{4}\neq-\frac{i(\widehat{\eta}z+\widehat{\tau}m)}{\sqrt{z^2-m^2-|\xi'|^2}}\quad\forall(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)},|\xi'|\leq R.$$

Squaring the equation, multiplying with $z^2-m^2-|\xi'|^2$ and setting $\widehat{d}=\widehat{\eta}^2-\widehat{\tau}^2$ yields

$$\left(1-\frac{\widehat{d}}{4}\right)^2(|\xi'|^2+m^2-z^2)\neq\widehat{\eta}^2z^2+2\widehat{\eta}\widehat{\tau}mz+\widehat{\tau}^2m^2\quad\forall(\widehat{\eta},\widehat{\tau})\in\overline{\operatorname{ran}(\eta,\tau)},|\xi'|\leq R. \tag{8.12}$$

By the solution formula for quadratic equations, this is equivalent to

$$\begin{aligned}
z&\neq\frac{-\widehat{\eta}\widehat{\tau}m\pm\sqrt{(\widehat{\eta}\widehat{\tau}m)^2+((1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2)((1-\frac{\widehat{d}}{4})^2(|\xi'|^2+m^2)-\widehat{\tau}^2m^2)}}{(1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2}\\
&=\frac{-\widehat{\eta}\widehat{\tau}m\pm\sqrt{((1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2)(1-\frac{\widehat{d}}{4})^2(|\xi'|^2+m^2)-(1-\frac{\widehat{d}}{4})^2\widehat{\tau}^2m^2}}{(1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2}\\
&=\frac{-\widehat{\eta}\widehat{\tau}m\pm(1-\frac{\widehat{d}}{4})\sqrt{((1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2)(|\xi'|^2+m^2)-\widehat{\tau}^2m^2}}{(1-\frac{\widehat{d}}{4})^2+\widehat{\eta}^2}
\end{aligned} \tag{8.13}$$

for all $|\xi'| \le R$ and $(\widehat{\eta}, \widehat{\tau}) \in \overline{\operatorname{ran}(\eta, \tau)}$. Using the identity $(1 - \frac{\widehat{d}}{4})^2 + \widehat{\eta}^2 = (1 + \frac{\widehat{d}}{4})^2 + \widehat{\tau}^2$ shows that the right-hand side of the above inequality equals

$$\frac{-\widehat{\eta}\widehat{\tau}m \pm (1 - \frac{\widehat{d}}{4})\sqrt{((1 + \frac{\widehat{d}}{4})^2 + \widehat{\tau}^2)(|\xi'|^2 + m^2) - \widehat{\tau}^2 m^2}}{(1 + \frac{\widehat{d}}{4})^2 + \widehat{\tau}^2}$$
$$= \frac{-\widehat{\eta}\widehat{\tau}m \pm (1 - \frac{\widehat{d}}{4})\sqrt{((1 + \frac{\widehat{d}}{4})^2 + \widehat{\tau}^2)|\xi'|^2 + (1 + \frac{d}{4})^2 m^2}}{(1 + \frac{\widehat{d}}{4})^2 + \widehat{\tau}^2}.$$

Thus, the expression in the square root is nonnegative and therefore the right-hand side in (8.13) is a real. Moreover, z is assumed to be in $\mathbb{C} \setminus \mathbb{R}$. Hence, (8.13) is valid, and therefore also (8.12) and (8.11) are true. Consequently, according to (8.10) we can choose $\underline{\varepsilon_{\mathrm{gr},1}} > 0$ sufficiently small such that

$$\begin{aligned} |p(\widehat{\eta}, \widehat{\tau}, \xi', \varepsilon)| &\ge \left|1 - \frac{\widehat{\eta}^2 - \widehat{\tau}^2}{4} + \frac{i(\widehat{\eta}z + \widehat{\tau}m)}{\sqrt{z^2 - m^2 - |\xi'|^2}}\right| - C\varepsilon R \\ &\ge \inf_{(\widehat{\eta}, \widehat{\tau}) \in \overline{\operatorname{ran}(\eta,\tau)}, |\xi'| \le R} \left|1 - \frac{\widehat{\eta}^2 - \widehat{\tau}^2}{4} + \frac{i(\widehat{\eta}z + \widehat{\tau}m)}{\sqrt{z^2 - m^2 - |\xi'|^2}}\right| - C\varepsilon R \\ &\ge \frac{\inf_{(\widehat{\eta}, \widehat{\tau}) \in \overline{\operatorname{ran}(\eta,\tau)}, |\xi'| \le R} \left|1 - \frac{\widehat{\eta}^2 - \widehat{\tau}^2}{4} + \frac{i(\widehat{\eta}z + \widehat{\tau}m)}{\sqrt{z^2 - m^2 - |\xi'|^2}}\right|}{2} > 0 \end{aligned}$$

for all $|\xi'| \le R$, $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{gr},1}})$ and $(\widehat{\eta}, \widehat{\tau}) \in \overline{\operatorname{ran}(\eta, \tau)}$. Combined with (8.9) this gives us (8.7), which completes the proof. $\square$

In the next statements we use similarly as in Section 5.1.2 the functions $\phi^a_{n'}$ and $\vartheta^a_{n'}$, where $a \in (0, \infty)$ and $n' \in \mathbb{Z}^{\theta-1}$, from Corollary A.3. They allow us to construct a uniformly bounded right inverse of $I + D^{\zeta,\kappa}_\varepsilon(z) Q^{\zeta,\kappa}_{\eta,\tau}$ with the help of the operators $(I + \underline{D^{\zeta_{x'_0},\kappa}_\varepsilon}(z) Q^{\zeta,\kappa}_{\eta,\tau}(x'_0))^{-1}$, $x'_0 \in \mathbb{R}^{\theta-1}$; cf. the text between Proposition 5.15 and Lemma 5.16.

Lemma 8.6. *Let $z \in \rho(H)$, $\varepsilon \in (0, \varepsilon_{\mathrm{tub}})$ and $a_\varepsilon = \varepsilon^{1/6}$. Then,*

$$(1 - \vartheta^{a_\varepsilon}_{n'}) \underline{E_\varepsilon}(z) \phi^{a_\varepsilon}_{n'} = (1 - \vartheta^{a_\varepsilon}_{n'}) \underline{D^{\zeta,\kappa}_\varepsilon}(z) \phi^{a_\varepsilon}_{n'}.$$

Proof. This follows from Lemma 5.16 and the fact that $\mathfrak{J}$ and $\mathfrak{J}^*$ commute with $\vartheta^{a_\varepsilon}_{n'}$ and $\phi^{a_\varepsilon}_{n'}$. $\square$

Proposition 8.7. *Let ζ and κ be as in (5.42), q be as in (5.1), $\eta, \tau \in C^1_b(\Sigma; \mathbb{R})$, $d = \eta^2 - \tau^2$ satisfy (8.4), $Q^{\zeta,\kappa}_{\eta,\tau}$ be defined by (5.18) and $z \in \mathbb{C} \setminus \mathbb{R}$. Then, there exists an $\underline{\varepsilon_{\mathrm{gr},2}} \in (0, \varepsilon_{ABC}]$, with $\varepsilon_{ABC} > 0$ chosen according to (4.19), such that $I + \underline{D^{\zeta,\kappa}_\varepsilon(z)} \overline{Q^{\zeta,\kappa}_{\eta,\tau}}$ has a right inverse which is uniformly bounded in $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ with respect to $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{gr},2}})$.*

Proof. We define for $\varepsilon \in (0, \min\{\varepsilon_{ABC}, \underline{\varepsilon_{\mathrm{gr},1}}\})$ with $\underline{\varepsilon_{\mathrm{gr},1}}$ chosen as in Proposition 8.5

$$\begin{aligned} &\underline{R_\varepsilon} : L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N), \\ &\underline{R_\varepsilon} := \sum_{n' \in \mathbb{Z}} \phi_{n'}^{a_\varepsilon} \underline{R_{n',\varepsilon}} \vartheta_{n'}^{a_\varepsilon}, \end{aligned}$$

with $a_\varepsilon = \varepsilon^{1/6}$ and $\underline{R_{n',\varepsilon}} := (I + \underline{D_\varepsilon^{\zeta_{a_\varepsilon n'}, \kappa}}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n'))^{-1}$. The equality $\vartheta_{n'}^{a_\varepsilon}\phi_{n'}^{a_\varepsilon} = \phi_{n'}^{a_\varepsilon}$, implies $\underline{R_\varepsilon} = \sum_{n' \in \mathbb{Z}^{\theta-1}} \vartheta_{n'}^{a_\varepsilon} \phi_{n'}^{a_\varepsilon} \underline{R_{n',\varepsilon}} \vartheta_{n'}^{a_\varepsilon}$ and therefore Proposition 8.5 and Corollary C.4 show that $\underline{R_\varepsilon}$ is well-defined and uniformly bounded by

$$\|\underline{R_\varepsilon}\|_{\underline{0} \to \underline{0}} \leq 11^{\theta-1} \sup_{n' \in \mathbb{Z}^{\theta-1}} \left\|(I + \underline{D_\varepsilon^{\zeta_{a_\varepsilon n'}, \kappa}}(z) Q_{\eta,\tau}^{\zeta,\kappa}(a_\varepsilon n'))^{-1}\right\|_{\underline{0} \to \underline{0}} \leq C,$$

where $C > 0$ does not depend on ε. In the exact same way as in the *Steps 1–3* of Proposition 5.17 one shows with the help of Proposition 8.4, Proposition 8.5 and Lemma 8.6 that

$$(I + \underline{D_\varepsilon^{\zeta,\kappa}}(z) Q_{\eta,\tau}^{\zeta,\kappa}) \underline{R_\varepsilon} = I + \underline{K_\varepsilon} + \underline{L_\varepsilon}, \tag{8.14}$$

where $\underline{K_\varepsilon}$ acts as a bounded operator from $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ and the norm estimates

$$\|\underline{K_\varepsilon}\|_{\underline{0} \to \underline{1}} \leq C \frac{1 + |\log(\varepsilon)|}{a_\varepsilon^2} \quad \text{and} \quad \|\underline{L_\varepsilon}\|_{\underline{0} \to \underline{0}} \leq C a_\varepsilon (1 + |\log(\varepsilon))$$

Moreover, (8.14) shows that the operator $\underline{K_\varepsilon} + \underline{L_\varepsilon}$ is uniformly bounded in $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ with respect to $\varepsilon \in (0, \min\{\varepsilon_{ABC}, \underline{\varepsilon_{\mathrm{gr},1}}\})$. According to Proposition 3.15 (iii) the operator $I + \mathcal{C}_z V = I + \underline{B_0}(z) V$ (with $V = \eta I_N + \tau\beta$) is continuously invertible in $L^2(\Sigma; \mathbb{C}^N)$ and $H^{1/2}(\Sigma; \mathbb{C}^N)$. Thus, by (5.11), (5.18), (8.2) and (8.5) the operator $(I + \underline{D_0^{\zeta,\kappa}}(z) Q_{\eta,\tau}^{\zeta,\kappa})^{-1}$ is continuously invertible in $L^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ and $H^{1/2}(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$. Using this observation as well as the properties of $\underline{L_\varepsilon}$ and $\underline{K_\varepsilon}$ one proves the assertion of the proposition in the same way as in *Step 4* of Proposition 5.17. □

Proposition 8.8. *Let Σ be a is C_b^2-graph as described in the beginning of Section 5.1.2, q be as in* (5.1)*, $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma; \mathbb{R})$ satisfy* (8.4) *and $z \in \mathbb{C} \setminus \mathbb{R}$. Then, there exists an $\underline{\varepsilon_{\mathrm{conv}}} \in (0, \varepsilon_{ABC}]$, with $\varepsilon_{ABC} > 0$ from* (4.19)*, such that $I + B_\varepsilon(z)V$ has a inverse which is uniformly bounded in $L^2(\Sigma; \mathbb{C}^N)$ with respect to $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{conv}}})$.*

Proof. The statement can be proven in the same way as Proposition 5.18. □

Finally, we are able to provide the main theorem of this section.

Theorem 8.9. *Let Σ be a special C^2-boundary as in Definition 2.1, q be as in* (5.1)*, $V = \eta I_N + \tau\beta$ with $\eta, \tau \in C_b^1(\Sigma;\mathbb{R})$ satisfy* (8.4) *and $z \in \mathbb{C} \setminus \mathbb{R}$. Then, the operator $H_{V\delta_\Sigma}$ is self-adjoint and there exists an $\underline{\varepsilon_{\mathrm{conv}}} > 0$ such that for any $r \in (0, \frac{1}{2})$ exists a $C > 0$ such that*

$$\big\|(H_{\underline{V_\varepsilon}} - z)^{-1} - (H_{V\delta_\Sigma} - z)^{-1}\big\|_{L^2(\mathbb{R}^\theta;\mathbb{C}^N)\to L^2(\mathbb{R}^\theta;\mathbb{C}^N)} \leq C\varepsilon^{1/2-r}$$

for $\varepsilon \in (0, \underline{\varepsilon_{\mathrm{conv}}})$. In particular, $H_{\underline{V_\varepsilon}}$ converges to $H_{V\delta_\Sigma}$ in the norm resolvent sense as $\varepsilon \to 0$.

Proof. In the case that Σ is a rotated C_b^2-graph the assertion follows from Theorem 8.3, the text below (8.4) and Proposition 8.8. In the general case one can use the C^1-partition of unity $\widehat{\varphi}_1, \dots, \widehat{\varphi}_p$ from Corollary A.5 and the same strategy as in the proof of Theorem 5.20 in order to prove the assertion. We remark that in the current semilocal case the choice of the partition of unity is crucial since the property $\widehat{\varphi}_l(x_\Sigma) = \widehat{\varphi}_l(x)$ for $x = x_\Sigma + t\nu(x_\Sigma)$ with $(x_\Sigma, t) \in \Sigma \times (-\frac{\varepsilon_{\mathrm{tub}}}{2}, \frac{\varepsilon_{\mathrm{tub}}}{2})$, which the C^1-partition of unity from Corollary A.5 possesses, guarantees that $\underline{V_\varepsilon}$ and $\widehat{\varphi}_l$ commute for $\varepsilon \in (0, \frac{\varepsilon_{\mathrm{tub}}}{2})$. Making use of this property allows one to prove formula (5.66) in the semilocal case in a very similar manner, which, in turn, ensures that the proof strategy from Theorem 5.20 is successful. $\square$

Appendix A. Partitions of unity

In this chapter, which contains results from [15], we construct various useful partitions of unity. Similarly as in [54, Chapter 3], we define a partition of unity as follows.

Definition A.1. *We call a sequence of functions $(\varphi_j)_{j\in J}$, where J is a countable (finite or infinite) index set and $\varphi_j \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, ($\varphi_j \in C^k(\mathbb{R}^n)$, $k, n \in \mathbb{N}$) for all $j \in J$, a partition (C^k-partition) of unity for a set $S \subset \mathbb{R}^n$ if the following three conditions are met:*

(i) $\phi_j(x) \geq 0$ *for* $x \in \mathbb{R}^n$ *and* $j \in J$.

(ii) *Every* $x \in \mathbb{R}^n$ *has a neighbourhood that intersects* $\operatorname{supp} \phi_j$ *for only finitely many* j*'s.*

(iii) $\sum_{j\in J} \varphi_j(x) = 1$ *for all* $x \in S$.

Moreover, if $(W_j)_{j\in J}$ is an open cover of S and $\operatorname{supp} \varphi_j \subset W_j$ for all $j \in J$, then we call $(\varphi_j)_{j\in J}$ a partition (C^k-partition) of unity for S subordinate to the open cover $(W_j)_{j\in J}$.

In this thesis we often use partitions of unity in settings where derivatives are involved; cf. Section 8.1 and the proofs of Proposition 5.17, Theorem 5.20 and Theorem 6.7. Thus, it is important to construct partitions of unity with uniformly bounded derivatives. We show in several situations that such a choice is possible, starting with the case $S = \mathbb{R}^n$, $n \in \{1, 2, 3\}$.

Proposition A.2. *Let $n \in \{1, 2, 3\}$. Then, there exists a partition of unity $(\phi_j)_{j\in\mathbb{Z}^n}$ for $\mathbb{R}^n$ subordinate to the open cover $(B(j, 1))_{j\in\mathbb{Z}^n}$ of $\mathbb{R}^n$ and a sequence of smooth functions $(\vartheta_j)_{j\in\mathbb{N}}$ which have the following properties:*

(i) $\sup_{j\in\mathbb{Z}^n} \max\{\|\phi_j\|_{W^1_\infty(\mathbb{R}^n)}, \|\vartheta_j\|_{W^1_\infty(\mathbb{R}^n)}\} < \infty$.

(ii) $\operatorname{supp} \vartheta_j \subset B(j, 3)$, $0 \leq \vartheta_j \leq 1$ *and* $\vartheta_j = 1$ *on* $B(j, 2)$ *for all* $j \in \mathbb{Z}^n$.

Proof. Note that since $\theta \in \{1, 2, 3\}$, the family $(B(j, 7/8))_{j\in\mathbb{Z}^n}$ is also an open cover of $\mathbb{R}^n$. We start by choosing a function $\phi \in C^\infty(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B(0, 7/8)$ and $\operatorname{supp} \phi \subset B(0, 1)$. Furthermore, we set $\widetilde{\phi}_j := \phi(\cdot - j)$ for $j \in \mathbb{Z}^n$. Then, $0 \leq \widetilde{\phi}_j \leq 1$, $\widetilde{\phi}_j = 1$ on $B(j, 7/8)$ and $\operatorname{supp} \widetilde{\phi}_j \subset B(j, 1)$. Next, we fix a bijection $\mathcal{Z} : \mathbb{N} \to \mathbb{Z}^n$ and set $\phi_{\mathcal{Z}(1)} := \widetilde{\phi}_{\mathcal{Z}(1)}$ and $\phi_{\mathcal{Z}(l)} := (1 - \widetilde{\phi}_{\mathcal{Z}(1)}) \cdot \cdots \cdot (1 - \widetilde{\phi}_{\mathcal{Z}(l-1)})\widetilde{\phi}_{\mathcal{Z}(l)}$

for $l \in \mathbb{N} \setminus \{1\}$. Then, $\operatorname{supp} \phi_j \subset \operatorname{supp} \widetilde{\phi}_j$, $0 \leq \widetilde{\phi}_j \leq 1$ for $j \in \mathbb{Z}^n$ and one gets via induction for $l \in \mathbb{N}$

$$\sum_{k=1}^{l} \phi_{\mathcal{Z}(k)} = 1 - \prod_{k=1}^{l} (1 - \widetilde{\phi}_{\mathcal{Z}(k)}).$$

This implies $\sum_{j \in \mathbb{Z}^n} \phi_j(x) = \sum_{l=1}^{\infty} \phi_{\mathcal{Z}(l)}(x) = 1$ for $x \in \mathbb{R}^n$. Thus, $(\phi_j)_{j \in \mathbb{Z}^n}$ is a partition of unity for $\mathbb{R}^n$ subordinate to $(B(j,1))_{j \in \mathbb{Z}^n}$.

Furthermore, let $l \in \mathbb{N}$, $w \in \{1, \dots, n\}$ and $x \in \mathbb{R}^n$. We estimate

$$\begin{aligned} |\partial_w \phi_{\mathcal{Z}(l)}(x)| =& \Big| \partial_w (\widetilde{\phi}_{\mathcal{Z}(l)}(x)) \prod_{k=1}^{l-1} (1 - \widetilde{\phi}_{\mathcal{Z}(k)}(x)) \\ & - \sum_{k=1}^{l-1} \widetilde{\phi}_{\mathcal{Z}(l)}(x) (\partial_w \widetilde{\phi}_{\mathcal{Z}(k)}(x)) \prod_{r=1, r \neq k}^{l-1} (1 - \widetilde{\phi}_{\mathcal{Z}(r)}(x)) \Big| \\ \leq& \sum_{k=1}^{l} \big| \partial_w \widetilde{\phi}_{\mathcal{Z}(k)}(x) \big| = \sum_{k=1, x \in B(\mathcal{Z}(k),1)}^{l} \big| \partial_w \widetilde{\phi}_{\mathcal{Z}(k)}(x) \big| \\ \leq& 2^n \|\partial_w \phi\|_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

where we used that $x \in \mathbb{R}^n$ can be contained in at most 2^n balls of the type $B(j,1)$ with $j \in \mathbb{Z}^n$. This shows that the derivatives of the ϕ_j's are uniformly bounded by $2^n \|\phi\|_{W^1_\infty(\mathbb{R}^n)}$. Next, we construct the sequence $(\vartheta_j)_{j \in \mathbb{Z}^n}$. To do so, we choose $\vartheta \in C^\infty(\mathbb{R}^n)$ such that $0 \leq \vartheta \leq 1$, $\theta = 1$ on $B(0,2)$ and $\operatorname{supp} \vartheta \subset B(0,3)$. Then, we define $\vartheta_j := \vartheta(\cdot - j)$. The constructed sequence has the claimed properties. □

A useful consequence of this proposition is the following corollary.

Corollary A.3. *Let $n \in \{1,2,3\}$ and $b > 0$. Then, for all $a \in (0,b)$ exists a partition of unity $(\phi_j^a)_{j \in \mathbb{Z}^n}$ for $\mathbb{R}^n$ subordinate to the open cover $(B(ja,a))_{j \in \mathbb{Z}^n}$ of $\mathbb{R}^n$ and a sequence of smooth functions $(\vartheta_j^a)_{j \in \mathbb{Z}^n}$ which have the following properties:*

(i) $\sup_{j \in \mathbb{Z}^n} \max\{\|\phi_j^a\|_{W^1_\infty(\mathbb{R}^n)}, \|\vartheta_j^a\|_{W^1_\infty(\mathbb{R}^n)}\} < \frac{C}{a}$, *where $C > 0$ does not depend on $a \in (0,b)$.*

(ii) $\operatorname{supp} \vartheta_j^a \subset B(ja, 3a)$, $0 \leq \vartheta_j^a \leq 1$ *and* $\vartheta_j^a = 1$ *on* $B(ja, 2)$ *for all* $j \in \mathbb{Z}^n$.

Proof. Define $\phi_j^a := \phi_j(\frac{\cdot}{a})$ and $\vartheta_j^a := \vartheta_j^a(\frac{\cdot}{a})$ for $j \in \mathbb{Z}^n$. Then, all the claims follow directly from Proposition A.4. □

Next, we also find suitable partitions of unity for special C^2-surfaces defined in Definition 2.1.

Proposition A.4. *Let $\Sigma \subset \mathbb{R}^\theta$, $\theta \in \{2,3\}$, be a special C^2-surface as in Definition 2.1 and $W_1, \dots, W_p$ be the corresponding open cover of Σ. Then, there exists a partition of unity $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta)$ for Σ subordinate to the open cover $W_1, \dots, W_p$ of Σ. Moreover, there exist functions $\chi_1, \dots, \chi_p \in C_b^\infty(\mathbb{R}^\theta)$ such that* $\operatorname{supp} \chi_l \subset W_l$ *and* $\varphi_l \chi_l = \varphi_l$ *for all* $l \in \{1, \dots, p\}$.

Proof. According to [69, Appendix 1, Lemma 1.2 and Lemma 1.3] there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^\theta$, $M \in \mathbb{N}$, $0 < \delta < \frac{\varepsilon_\Sigma}{4}$, with ε_Σ from Definition 2.1, and a sequence of real-valued C^∞-functions $(\phi_n)_{n\in\mathbb{N}}$ such that $(B(x_n, \delta))_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}^\theta$, $(\phi_n)_{n\in\mathbb{N}}$ is a partition of unity for $\mathbb{R}^\theta$, $\operatorname{supp} \phi_n \subset B(x_n, \delta)$ for all $n \in \mathbb{N}$, every point $x \in \mathbb{R}^\theta$ is contained in at most M of the sets $B(x_n, \delta)$, and the derivatives (of any order) of the functions ϕ_n are uniformly bounded. Next, we define the set $Y := \{x_n : B(x_n, 2\delta) \cap \Sigma \neq \emptyset\}$. Note that for all $x_n \in Y$ there exists an $l \in \{1, \dots, p\}$ such that $B(x_n, 2\delta) \subset W_l$. In fact, as $B(x_n, 2\delta) \cap \Sigma \neq \emptyset$, there exists $y_\Sigma \in B(x_n, 2\delta) \cap \Sigma$ and thus, Definition 2.1 implies $B(y_\Sigma, \varepsilon_\Sigma) \subset W_l$ for an $l \in \{1, \dots, p\}$. Hence, for any $y \in B(x_n, 2\delta)$ one has

$$|y - y_\Sigma| \leq |y - x_n| + |x_n - y_\Sigma| < 4\delta < \varepsilon_\Sigma,$$

which shows $B(x_n, 2\delta) \subset W_l$. Define the sets $I_1 := \{n : x_n \in Y, B(x_n, 2\delta) \subset W_1\}$ and $I_l := \{n : x_n \in Y, B(x_n, 2\delta) \subset W_l, B(x_n, 2\delta) \not\subset W_k, k \in \{1, \dots, l-1\}\}$ for $l \in \{2, \dots, p\}$. Then, it is not difficult to see that

$$\varphi_l = \sum_{n \in I_l} \phi_n \qquad \square$$

is a partition of unity having the claimed properties. Moreover, the construction of φ_l, $l \in \{1, \dots, p\}$, also implies $\operatorname{supp} \varphi_l + B(0, \delta) \subset W_l$. Thus, [54, Theorem 3.6] guarantees the existence of the functions $\chi_1, \dots, \chi_p \in C_b^\infty(\mathbb{R}^\theta)$ with $\operatorname{supp} \chi_l \subset W_l$ and $\varphi_l \chi_l = \varphi_l$ for $l \in \{1, \dots, p\}$.

Corollary A.5. *Let $\Sigma \subset \mathbb{R}^\theta$, $\theta \in \{2,3\}$, be a special C^2-surface as in Definition 2.1, $W_1, \dots, W_p$ be the corresponding open cover of Σ, Ω_ε be as in Definition 2.7 and $\varepsilon_{\text{tub}} > 0$ be chosen as in Proposition 2.12. Then, there exists a C^1-partition of unity $\widehat{\varphi}_1, \dots, \widehat{\varphi}_p \in C_b^1(\mathbb{R}^\theta)$ for $\Omega_{\frac{\varepsilon_{\text{tub}}}{2}}$ and a function $\varpi \in C^1(\mathbb{R})$ with $0 \leq \varpi \leq 1$ such that $\widehat{\varphi}_l(x_\Sigma + t\nu(x_\Sigma)) = \widehat{\varphi}_l(x_\Sigma)\varpi(t)$ for $(x_\Sigma, t) \in \Sigma \times (-\varepsilon_{\text{tub}}, \varepsilon_{\text{tub}})$, $\operatorname{supp} \widehat{\varphi}_l \cap \Sigma \subset W_l$ and $\operatorname{supp} \widehat{\varphi}_l \subset \Omega_{\varepsilon_{\text{tub}}}$ for all $l \in \{1, \dots, p\}$.*

Proof. Let $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta)$ be the partition of unity from Proposition A.4 and define for $l \in \{1, \dots, p\}$ the function $\widehat{\varphi}_l$ as the extension of $\varphi_l \restriction \Sigma$ given by Lemma 4.3. Then, the functions $\widehat{\varphi}_1, \dots, \widehat{\varphi}_p$ have the claimed properties because of the way they are constructed in Lemma 4.3. $\square$

Appendix B. Proof of the estimate (4.34)

Before we start with the proof of (4.34), let us mention that this section can be found in [14, Appendix B]. Let us shortly recall the problem setting. Let $z \in \rho(H)$ and $\widetilde{B}_\varepsilon(z) : \mathcal{B}^0(\Sigma) \to \mathcal{B}^0(\Sigma)$ and $\overline{B}_\varepsilon(z) : \mathcal{B}^{1/2}(\Sigma) \to \mathcal{B}^{1/2}(\Sigma)$ be the operators defined by (4.27) and (4.29), respectively. In this chapter we prove (4.34), i.e. we show that $\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)$ can be extended to a bounded operator from $\mathcal{B}^0(\Sigma)$ to $\mathcal{B}^{1/2}(\Sigma)$ and that

$$\|\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\|_{0\to 1/2} \leq C\varepsilon^{1/2}(1 + |\log(\varepsilon)|)^{1/2} \tag{B.1}$$

for some $C > 0$, which is used in (4.34) in *Step 3* in the proof of Proposition 4.10. With (4.28) and (4.32) one obtains for $f \in \mathcal{B}^{1/2}(\Sigma)$

$$\begin{aligned}\big(\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)\big) f(t)(x_\Sigma) = \int_{-1}^{1}\int_\Sigma \big(&G_z(x_\Sigma - y_\Sigma + \varepsilon t\nu(x_\Sigma) - \varepsilon s\nu(y_\Sigma)) \\ &- G_z(x_\Sigma - y_\Sigma + \varepsilon(t-s)\nu(x_\Sigma))\big) f(s)(y_\Sigma)\, d\sigma(y_\Sigma)\, ds\end{aligned} \tag{B.2}$$

for a.e. $t \in (-1,1)$ and for σ-a.e. $x_\Sigma \in \Sigma$, where G_z is the integral kernel of $R_z = (H-z)^{-1}$; cf. (3.3)–(3.4). Thus, in order to show (B.1), we proceed as follows: We prove in Proposition B.2 that for fixed $t \neq s \in (-1,1)$ the operator formally acting on $\psi \in L^2(\Sigma;\mathbb{C}^N)$ as

$$\begin{aligned}b_{t,s,\varepsilon}(z)\psi(x_\Sigma) = \int_\Sigma \big(&G_z(x_\Sigma - y_\Sigma + \varepsilon t\nu(x_\Sigma) - \varepsilon s\nu(y_\Sigma)) \\ &- G_z(x_\Sigma - y_\Sigma + \varepsilon(t-s)\nu(x_\Sigma))\big)\psi(y_\Sigma)\, d\sigma(y_\Sigma),\end{aligned} \tag{B.3}$$

$x_\Sigma \in \Sigma$, gives rise to a bounded operator from $L^2(\Sigma;\mathbb{C}^N)$ to $H^{1/2}(\Sigma;\mathbb{C}^N)$ and we prove an estimate for its operator norm. Then, we show in Lemma B.3 that the map $(s,t) \mapsto b_{t,s,\varepsilon}(z)$ is measurable and use (B.2) to transfer the results from $b_{t,s,\varepsilon}(z)$ to $\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)$.

In the following, we always assume $\varepsilon \in (0, \varepsilon_{ABC})$ with $\varepsilon_{ABC} > 0$ satisfying (4.19). Recall that ε_{tub} and ε_ι are the numbers that are specified in Proposition 2.12 and Proposition 2.9, respectively. Since $\varepsilon_{\text{tub}} < \varepsilon_\iota$, see the proof of Proposition 2.12, we conclude from (4.19) that $\varepsilon_{ABC} < \frac{\varepsilon_\iota}{2}$. We define for $t \neq s \in (-1,1)$ and $x_\Sigma, y_\Sigma \in \Sigma$

$$\begin{aligned}\Delta G_z(x_\Sigma, y_\Sigma, t, s) := {}&G_z(x_\Sigma - y_\Sigma + \varepsilon t\nu(x_\Sigma) - \varepsilon s\nu(y_\Sigma)) \\ &- G_z(x_\Sigma - y_\Sigma + \varepsilon(t-s)\nu(x_\Sigma)).\end{aligned} \tag{B.4}$$

Moreover, we introduce for $t \neq s \in (-1,1)$ and $x_\Sigma, y_\Sigma \in \Sigma$ the quantities

$$\begin{aligned} z_0(x_\Sigma, y_\Sigma, t, s) &:= x_\Sigma - y_\Sigma + \varepsilon t \nu(x_\Sigma) - \varepsilon s \nu(y_\Sigma) = \iota(x_\Sigma, \varepsilon t) - \iota(y_\Sigma, \varepsilon s), \\ z_1(x_\Sigma, y_\Sigma, t, s) &:= x_\Sigma - y_\Sigma + \varepsilon(t-s)\nu(x_\Sigma) = \iota(x_\Sigma, \varepsilon(t-s)) - \iota(y_\Sigma, 0), \\ z_\mu(x_\Sigma, y_\Sigma, t, s) &:= \mu z_0(x_\Sigma, y_\Sigma, t, s) + (1-\mu) z_1(x_\Sigma, y_\Sigma, t, s) \\ &= \iota(x_\Sigma, \mu\varepsilon t + (1-\mu)\varepsilon(t-s)) - \iota(y_\Sigma, \mu\varepsilon s) \quad \text{for } \mu \in (0,1), \\ L(x_\Sigma, y_\Sigma, t, s) &:= |x_\Sigma - y_\Sigma| + |\varepsilon(t-s)|. \end{aligned}$$

Then, $\Delta G_z(x_\Sigma, y_\Sigma, t, s) = G_z(z_0(x_\Sigma, y_\Sigma, t, s)) - G_z(z_1(x_\Sigma, y_\Sigma, t, s))$. It follows from Proposition 2.9 (ii) that for $\mu \in [0,1]$ the inequalities

$$C_{\iota,2}^{-1} L(x_\Sigma, y_\Sigma, t, s) \leq |z_\mu(x_\Sigma, y_\Sigma, t, s)| \leq C_{\iota,2} L(x_\Sigma, y_\Sigma, t, s) \tag{B.5}$$

hold. To shorten notation we also set

$$c := \frac{C_{G,2}}{C_{\iota,2}} > 0 \tag{B.6}$$

with $C_{G,2}$ from Proposition 3.4. Furthermore, until Lemma B.3 we fix $t \neq s \in (-1,1)$ and hence omit the arguments t, s in the functions L, ΔG_z, z_0, z_μ and z_1.

Lemma B.1. *Let G_z be the integral kernel of R_z in* (3.3)–(3.4)*, ΔG_z as in* (B.4)*, $l \in \{1, \dots, p\}$, and $\varkappa_l$ as in* (2.1)*. Then, the following is true:*

(i) *There exists $C > 0$ which does not depend on ε, t, and s such that*

$$|\Delta G_z(x_\Sigma, y_\Sigma)| \leq C \varepsilon L(x_\Sigma, y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma, y_\Sigma)}$$

for all $x_\Sigma, y_\Sigma \in \Sigma$.

(ii) *There exists $C > 0$ which does not depend on ε, t, and s such that*

$$\left| \frac{d}{dx'_k} \Delta G_z(\varkappa_l(x'), y_\Sigma) \right| \leq C \varepsilon L(\varkappa_l(x'), y_\Sigma)^{-\theta} e^{-cL(\varkappa_l(x'), y_\Sigma)}$$

for all $k \in \{1, \dots, \theta - 1\}$, $y_\Sigma \in \Sigma$, and $x' \in \varkappa_l^{-1}(\Sigma)$.

Proof. Before we prove (i) and (ii), we show a useful estimate of the difference

$z_0(x_\Sigma, y_\Sigma) - z_1(x_\Sigma, y_\Sigma)$. Since $\varepsilon_{ABC} < \frac{\varepsilon_\iota}{2}$, it follows from Proposition 2.9 (ii) that

$$\begin{aligned}
|z_0(x_\Sigma, y_\Sigma) - z_1(x_\Sigma, y_\Sigma)| &= |\varepsilon t\nu(x_\Sigma) - \varepsilon s\nu(y_\Sigma) - \varepsilon(t-s)\nu(x_\Sigma)| \\
&= \varepsilon|s||\nu(x_\Sigma) - \nu(y_\Sigma)| \\
&\leq \frac{\varepsilon}{\varepsilon_{ABC}}|\varepsilon_{ABC}\nu(x_\Sigma) - \varepsilon_{ABC}\nu(y_\Sigma)| \\
&\leq \frac{\varepsilon}{\varepsilon_{ABC}}\big(|x_\Sigma - y_\Sigma| + \big|x_\Sigma + \varepsilon_{ABC}\nu(x_\Sigma) - y_\Sigma - \varepsilon_{ABC}\nu(y_\Sigma)\big|\big) \\
&= \frac{\varepsilon}{\varepsilon_{ABC}}\big(|x_\Sigma - y_\Sigma| + |\iota(x_\Sigma, \varepsilon_{ABC}) - \iota(y_\Sigma, \varepsilon_{ABC})|\big) \\
&\leq \frac{1 + C_{\iota,2}}{\varepsilon_{ABC}}\varepsilon|x_\Sigma - y_\Sigma| \\
&\leq \frac{1 + C_{\iota,2}}{\varepsilon_{ABC}}\varepsilon L(x_\Sigma, y_\Sigma)
\end{aligned} \tag{B.7}$$

for all $x_\Sigma, y_\Sigma \in \Sigma$.

(i) Applying Lemma 2.8, Proposition 3.4, (B.5) and (B.7) yields

$$\begin{aligned}
|\Delta G_z(x_\Sigma, y_\Sigma)| &= |G_z(z_0(x_\Sigma, y_\Sigma)) - G_z(z_1(x_\Sigma, y_\Sigma))| \\
&\leq C \sup_{\mu\in[0,1], j\in\{1,\dots,\theta\}} |\partial_j G_z(z_\mu(x_\Sigma, y_\Sigma))||z_0(x_\Sigma, y_\Sigma) - z_1(x_\Sigma, y_\Sigma)| \\
&\leq C \sup_{\mu\in[0,1], j\in\{1,\dots,\theta\}} |\partial_j G_z(z_\mu(x_\Sigma, y_\Sigma))|\varepsilon L(x_\Sigma, y_\Sigma) \\
&\leq C \sup_{\mu\in[0,1]} |z_\mu(x_\Sigma, y_\Sigma)|^{-\theta} e^{-C_{G,2}|z_\mu(x_\Sigma, y_\Sigma)|}\varepsilon L(x_\Sigma, y_\Sigma) \\
&\leq C\varepsilon L(x_\Sigma, y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma, y_\Sigma)}
\end{aligned}$$

for all $x_\Sigma, y_\Sigma \in \Sigma$, where c is defined in (B.6) and $C > 0$ is a constant which does not depend on ε, t, and s. Hence, the claim in (i) is shown.

(ii) For $k \in \{1, \dots, \theta - 1\}$, $y_\Sigma \in \Sigma$, and $x' \in \varkappa_l^{-1}(\Sigma)$ we compute

$$\begin{aligned}
&\frac{d}{dx'_k} \Delta G_z(\varkappa_l(x'), y_\Sigma) = \frac{d}{dx'_k}\big(G_z(z_0(\varkappa_l(x'), y_\Sigma)) - G_z(z_1(\varkappa_l(x'), y_\Sigma))\big) \\
&= \sum_{j=1}^{\theta} (\partial_j G_z)(z_0(\varkappa_l(x'), y_\Sigma)) \frac{d}{dx'_k}(z_0(\varkappa_l(x'), y_\Sigma))[j] \\
&\quad - \sum_{j=1}^{\theta} (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma)) \frac{d}{dx'_k}(z_1(\varkappa_l(x'), y_\Sigma))[j] \\
&= \sum_{j=1}^{\theta} \big((\partial_j G_z)(z_0(\varkappa_l(x'), y_\Sigma)) - (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma))\big) \frac{d}{dx'_k}(z_0(\varkappa_l(x'), y_\Sigma))[j] \\
&\quad + \sum_{j=1}^{\theta} (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma)) \frac{d}{dx'_k}\left(z_0(\varkappa_l(x'), y_\Sigma) - z_1(\varkappa_l(x'), y_\Sigma)\right)[j] \\
&= \sum_{j=1}^{\theta} \big((\partial_j G_z)(z_0(\varkappa_l(x'), y_\Sigma)) - (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma))\big) \frac{d}{dx'_k}(z_0(\varkappa_l(x'), y_\Sigma))[j] \\
&\quad + \sum_{j=1}^{\theta} \varepsilon s (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma)) \frac{d}{dx'_k}(\nu_l(x'))[j],
\end{aligned}$$

where the convention $\nu_l(x') = \nu(\varkappa_l(x'))$ was used in the last step. To estimate the second sum we use (B.5), Proposition 3.4 and $\zeta_l \in C_b^2(\mathbb{R}^{\theta-1}; \mathbb{C}^N)$ and obtain

$$\begin{aligned}
&\Big|\sum_{j=1}^{\theta} \varepsilon s (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma)) \frac{d}{dx'_k}(\nu_l(x'))[j]\Big| \\
&\qquad\qquad \le C\varepsilon \sup_{j \in \{1,\dots,\theta\}} |(\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma))| \|D\nu_l\|_{L^\infty(\mathbb{R}^{\theta-1}; \mathbb{R}^{\theta \times (\theta-1)})} \\
&\qquad\qquad \le C\varepsilon L(\varkappa_l(x'), y_\Sigma)^{-\theta} e^{-cL(\varkappa_l(x'), y_\Sigma)}.
\end{aligned}$$

For the remaining part given by

$$\sum_{j=1}^{\theta} \big((\partial_j G_z)(z_0(\varkappa_l(x'), y_\Sigma)) - (\partial_j G_z)(z_1(\varkappa_l(x'), y_\Sigma))\big) \frac{d}{dx'_k}(z_0(\varkappa_l(x'), y_\Sigma))[j] \tag{B.8}$$

we proceed in the same way as in the proof of (i). Using Lemma 2.8 as well as $\zeta_l \in C_b^2(\mathbb{R}^{\theta-1}; \mathbb{R})$ one can show that the absolute value of the expression in (B.8) is bounded by the term

$$C \sup_{\mu \in [0,1],\, n, j \in \{1,\dots,\theta\}} |\partial_n \partial_j G_z(z_\mu(\varkappa_l(x'), y_\Sigma))| \varepsilon L(\varkappa_l(x'), y_\Sigma),$$

which, in turn, is according to Proposition 3.4 and (B.5) also bounded by

$$C\varepsilon L(\varkappa_l(x'), y_\Sigma)^{-\theta} e^{-cL(\varkappa_l(x'),y_\Sigma)}. \qquad \square$$

To estimate the operator norm of $b_{t,s,\varepsilon}(z)$ in Proposition B.2 below we make use of a partition of unity $\varphi_1, \dots, \varphi_p$ for Σ subordinate to $W_1, \dots, W_p$ with the additional property that the derivatives are uniformly bounded; the existence of such a partition of unity is shown in Proposition A.4.

Proposition B.2. *Let $t \neq s \in (-1,1)$ and $\varepsilon \in (0, \varepsilon_{ABC})$. Then,* (B.3) *gives rise to a bounded operator $b_{t,s,\varepsilon}(z) : L^2(\Sigma; \mathbb{C}^N) \to H^{1/2}(\Sigma; \mathbb{C}^N)$ and there exists a $C > 0$ which does not depend on ε, t, and s such that*

$$\|b_{t,s,\varepsilon}(z)\|_{L^2(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)} \leq C\big(\varepsilon(1 + |\log(\varepsilon|t-s|)|)\big)^{1/2} \frac{1}{|t-s|^{1/2}}. \tag{B.9}$$

Proof. We split this proof into four steps. In *Step 1* we verify the preliminary estimate

$$\sup_{x_\Sigma \in \Sigma} \int_\Sigma L(x_\Sigma, y_\Sigma)^j e^{-cL(x_\Sigma,y_\Sigma)}\, d\sigma(y_\Sigma) \leq C \begin{cases} 1 + |\log(\varepsilon|t-s|)|, & j = 1-\theta, \\ \frac{1}{\varepsilon|t-s|}, & j = -\theta, \end{cases} \tag{B.10}$$

which will be used in *Step 2* and *Step 3* to obtain bounds for $b_{t,s,\varepsilon}(z)$ viewed as an operator from $L^2(\Sigma; \mathbb{C}^N)$ to $L^2(\Sigma; \mathbb{C}^N)$ and from $L^2(\Sigma; \mathbb{C}^N)$ to $H^1(\Sigma; \mathbb{C}^N)$, respectively. Finally, we conclude with an interpolation argument (B.9) in *Step 4*.

Step 1. Let $x_\Sigma \in \Sigma$ and $j \in \{1-\theta, -\theta\}$. Recall that Σ satisfies Definition 2.1 and let $\varphi_1, \dots, \varphi_p \in C_b^\infty(\mathbb{R}^\theta)$ be the partition of unity from Proposition A.4. Using the definition of the boundary integral, we can write

$$\begin{aligned} &\int_\Sigma L(x_\Sigma, y_\Sigma)^j e^{-cL(x_\Sigma,y_\Sigma)}\, d\sigma(y_\Sigma) \\ &\qquad = \sum_{n=1}^p \int_{\varkappa_n^{-1}(\Sigma)} L(x_\Sigma, \varkappa_n(y'))^j e^{-cL(x_\Sigma,\varkappa_n(y'))} \varphi_n(\varkappa_n(y')) \sqrt{1 + |\nabla\zeta_n(y')|^2}\, dy'. \end{aligned}$$

Hence, $0 \leq \varphi_n \leq 1$, $\zeta_n \in C_b^2(\mathbb{R}^{\theta-1}; \mathbb{R})$, and $\varkappa_n^{-1}(\Sigma) \subset \mathbb{R}^{\theta-1}$ yield

$$\begin{aligned} \int_\Sigma L(x_\Sigma, y_\Sigma)^j e^{-cL(x_\Sigma,y_\Sigma)}\, d\sigma(y_\Sigma) &\leq C \max_{n\in\{1,\dots,p\}} \int_{\mathbb{R}^{\theta-1}} L(x_\Sigma, \varkappa_n(y'))^j e^{-cL(x_\Sigma,\varkappa_n(y'))}\, dy' \\ &\leq C \max_{n\in\{1,\dots,p\}} \int_{\mathbb{R}^{\theta-1}} c^j L(x_\Sigma, \varkappa_n(y'))^j e^{-cL(x_\Sigma,\varkappa_n(y'))}\, dy', \end{aligned}$$

where $C > 0$ does not depend on x_Σ, t, s, and ε. Next, let $n \in \{1, \dots, p\}$ and fix $x'_n \in \mathbb{R}^{\theta-1}$ such that $|x_\Sigma - \varkappa_n(x'_n)| = \min_{y' \in \mathbb{R}^{\theta-1}} |x_\Sigma - \varkappa_n(y')|$. With this choice we obtain for all $y' \in \mathbb{R}^{\theta-1}$

$$\begin{aligned} \frac{1}{2}|x'_n - y'| &\leq \frac{1}{2}|\varkappa_n(x'_n) - \varkappa_n(y')| \\ &\leq \frac{1}{2}\big(|x_\Sigma - \varkappa_n(y')| + |x_\Sigma - \varkappa_n(x'_n)|\big) \\ &\leq |x_\Sigma - \varkappa_n(y')|. \end{aligned}$$

This implies for any $y' \in \mathbb{R}^{\theta-1}$

$$cL(x_\Sigma, \varkappa_n(y')) = c\big(|x_\Sigma - \varkappa_n(y')| + \varepsilon|t-s|\big) \geq \frac{c}{2}|x'_n - y'| + c\varepsilon|t-s|.$$

Moreover, $a \mapsto a^j e^{-a}$, $a > 0$, $j \in \{1-\theta, -\theta\}$, is a monotonically decreasing function and therefore we get with $\rho(x'_n, y') := \frac{c}{2}|x'_n - y'|$

$$\begin{aligned} &\int_\Sigma c^j L(x_\Sigma, y_\Sigma)^j e^{-cL(x_\Sigma, y_\Sigma)}\, d\sigma(y_\Sigma) \\ &\qquad \leq C \max_{n \in \{1,\dots,p\}} \int_{\mathbb{R}^{\theta-1}} (\rho(x'_n, y') + c\varepsilon|t-s|)^j e^{-\rho(x'_n, y') - c\varepsilon|t-s|}\, dy' \\ &\qquad \leq C \int_0^\infty (\rho + c\varepsilon|t-s|)^j e^{-\rho - c\varepsilon|t-s|} \rho^{\theta-2} d\rho \\ &\qquad \leq C \int_0^\infty (\rho + c\varepsilon|t-s|)^{j+\theta-2} e^{-\rho - c\varepsilon|t-s|}\, d\rho \\ &\qquad = C \int_{c\varepsilon|t-s|}^\infty \rho^{j+\theta-2} e^{-\rho}\, d\rho \\ &\qquad \leq C \begin{cases} 1 + |\log(\varepsilon|t-s|)|, & j = 1-\theta, \\ \frac{1}{\varepsilon|t-s|}, & j = -\theta, \end{cases} \end{aligned}$$

where $C > 0$ does not depend on x_Σ, t, s, and ε. This proves (B.10).

Step 2. In this step we verify the estimate

$$\|b_{t,s,\varepsilon}(z)\psi\|_{L^2(\Sigma;\mathbb{C}^N)} \leq C\varepsilon(1 + |\log(\varepsilon|t-s|)|)\|\psi\|_{L^2(\Sigma;\mathbb{C}^N)}, \quad \psi \in L^2(\Sigma;\mathbb{C}^N). \quad \text{(B.11)}$$

In fact, with the help of the Cauchy-Schwarz inequality, Lemma B.1 (i), and (B.10) we obtain for $\psi \in L^2(\Sigma;\mathbb{C}^N)$ and $x_\Sigma \in \Sigma$

$$\begin{aligned}
\big|b_{t,s,\varepsilon}(z)\psi(x_\Sigma)\big|^2 &= \Big|\int_\Sigma \Delta G_z(x_\Sigma,y_\Sigma)\psi(y_\Sigma)\,d\sigma(y_\Sigma)\Big|^2 \\
&\leq \int_\Sigma |\Delta G_z(x_\Sigma,y_\Sigma)|\,d\sigma(y_\Sigma) \int_\Sigma |\Delta G_z(x_\Sigma,y_\Sigma)||\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma) \\
&\leq C\varepsilon^2 \int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma)}\,d\sigma(y_\Sigma) \\
&\qquad \cdot \int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma)} |\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma) \\
&\leq C\varepsilon^2(1+|\log(\varepsilon|t-s|)|) \int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma)} |\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma).
\end{aligned} \tag{B.12}$$

Now, Fubini's theorem and (B.10) show

$$\begin{aligned}
&\int_\Sigma |b_{t,s,\varepsilon}(z)\psi(x_\Sigma)|^2\,d\sigma(x_\Sigma) \\
&\quad\leq C\varepsilon^2(1+|\log(\varepsilon|t-s|)|) \int_\Sigma\int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma)} |\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma)\,d\sigma(x_\Sigma) \\
&\quad= C\varepsilon^2(1+|\log(\varepsilon|t-s|)|) \int_\Sigma\int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma)}\,d\sigma(x_\Sigma) |\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma) \\
&\quad\leq C\varepsilon^2(1+|\log(\varepsilon|t-s|)|)^2 \int_\Sigma |\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma),
\end{aligned}$$

which yields (B.11).

Step 3. Next, we prove the estimate

$$\|b_{t,s,\varepsilon}(z)\psi\|_{H^1(\Sigma;\mathbb{C}^N)} \leq C\frac{1}{|t-s|}\|\psi\|_{L^2(\Sigma;\mathbb{C}^N)}, \quad \psi \in L^2(\Sigma;\mathbb{C}^N). \tag{B.13}$$

Let $\psi \in L^2(\Sigma;\mathbb{C}^N)$ and $\varkappa_l(x') = x_\Sigma \in \Sigma$ with $x' \in \mathbb{R}^{\theta-1}$. By Proposition A.4 the function φ_l and its derivatives are bounded. Thus, with $\widetilde{\varphi}_l := \varphi_l \circ \varkappa_l$ we have

$$\begin{aligned}
&\Big|\frac{d}{dx'_k}\Big(\widetilde{\varphi}_l(x')(b_{t,s,\varepsilon}(z)\psi)(\varkappa_l(x'))\Big)\Big|^2 \\
&\qquad\leq 2|\widetilde{\varphi}_l(x')\frac{d}{dx'_k}(b_{t,s,\varepsilon}(z)\psi)(\varkappa_l(x'))|^2 + 2\Big|\Big(\frac{d}{dx'_k}\widetilde{\varphi}_l(x')\Big)b_{t,s,\varepsilon}(z)\psi(\varkappa_l(x'))\Big|^2 \\
&\qquad\leq C\Big(\Big|\frac{d}{dx'_k}(b_{t,s,\varepsilon}(z)\psi)(\varkappa_l(x'))\Big|^2 + |b_{t,s,\varepsilon}(z)\psi(\varkappa_l(x'))|\Big).
\end{aligned}$$

Using the dominated convergence theorem and the properties of ΔG_z stated in Lemma B.1 one obtains

$$\frac{d}{dx'_k}(b_{t,s,\varepsilon}(z)\psi)(\varkappa_l(x')) = \int_\Sigma \frac{d}{dx'_k}\Delta G_z(\varkappa_l(x'),y_\Sigma)\psi(y_\Sigma)\,d\sigma(y_\Sigma).$$

Hence, we get with the Cauchy-Schwarz inequality, Lemma B.1 (ii), $x_\Sigma = \varkappa_l(x')$, and (B.10)

$$\begin{aligned}
\Big|\frac{d}{dx'_k}(b_{t,s,\varepsilon}(z)\psi)(\varkappa_l(x'))\Big|^2 &= \Big|\int_\Sigma \frac{d}{dx'_k}\Delta G_z(\varkappa_l(x'),y_\Sigma)\psi(y_\Sigma)\,d\sigma(y_\Sigma)\Big|^2 \\
&\leq \int_\Sigma\Big|\frac{d}{dx'_k}\Delta G_z(\varkappa_l(x'),y_\Sigma)\Big|\,d\sigma(y_\Sigma)\int_\Sigma\Big|\frac{d}{dx'_k}\Delta G_z(\varkappa_l(x'),y_\Sigma)\Big||\psi(y_\Sigma)|^2 d\sigma(y_\Sigma) \\
&\leq C\varepsilon^2\int_\Sigma L(x_\Sigma,y_\Sigma)^{-\theta}e^{-cL(x_\Sigma,y_\Sigma)}\,d\sigma(y_\Sigma) \\
&\qquad\cdot\int_\Sigma L(x_\Sigma,y_\Sigma)^{-\theta}e^{-cL(x_\Sigma,y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma) \\
&\leq C\frac{\varepsilon}{|t-s|}\int_\Sigma L(x_\Sigma,y_\Sigma)^{-\theta}e^{-cL(x_\Sigma,y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma).
\end{aligned}$$

According to (B.12) we can estimate

$$|b_{t,s,\varepsilon}(z)\psi(\varkappa_l(x'))|^2 \leq C\varepsilon^2(1+|\log(\varepsilon|t-s|)|)\int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta}e^{-cL(x_\Sigma,y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma),$$

where $x_\Sigma = \varkappa_l(x')$. Moreover, $1+|\log(a)| \leq C\frac{1}{a}$ for $a \in (0, 2\varepsilon_{ABC})$ yields

$$|b_{t,s,\varepsilon}(z)\psi(\varkappa_l(x'))|^2 \leq C\frac{\varepsilon}{|t-s|}\int_\Sigma L(x_\Sigma,y_\Sigma)^{1-\theta}e^{-cL(x_\Sigma,y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma).$$

Thus,

$$\begin{aligned}
&\int_{\varkappa_l^{-1}(\Sigma)}\Big|\frac{d}{dx'_k}\Big(\varphi_l(\varkappa_l(x'))b_{t,s,\varepsilon}(z)\psi\Big)(x')\Big|^2\,dx' \\
&\leq C\frac{\varepsilon}{|t-s|}\int_{\varkappa_l^{-1}(\Sigma)}\int_\Sigma(L(\varkappa_l(x'),y_\Sigma)^{1-\theta}+L(\varkappa_l(x'),y_\Sigma)^{-\theta}) \\
&\qquad\qquad\cdot e^{-cL(\varkappa_l(x'),y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma)\,dx' \\
&\leq C\frac{\varepsilon}{|t-s|}\int_{\varkappa_l^{-1}(\Sigma)}\int_\Sigma(L(\varkappa_l(x'),y_\Sigma)^{1-\theta}+L(\varkappa_l(x'),y_\Sigma)^{-\theta}) \\
&\qquad\qquad\cdot e^{-cL(\varkappa_l(x'),y_\Sigma)}|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma)\sqrt{1+|\nabla\zeta_l(x')|^2}\,dx' \\
&\leq C\frac{\varepsilon}{|t-s|}\int_\Sigma\int_\Sigma(L(\varkappa_l(x'),y_\Sigma)^{1-\theta}+L(x_\Sigma,y_\Sigma)^{-\theta})e^{-cL(x_\Sigma,y_\Sigma)}\,|\psi(y_\Sigma)|^2\,d\sigma(y_\Sigma)d\sigma(x_\Sigma).
\end{aligned}$$

Therefore, Fubini's theorem, (B.10) and using $1+|\log(a)| \leq C\frac{1}{a}$ for $a \in (0, 2\varepsilon_{ABC})$ again yield

$$\int_{\varkappa_l^{-1}(\Sigma)}\Big|\frac{d}{dx'_k}\Big(\varphi_l(\varkappa_l(x'))b_{t,s,\varepsilon}(z)\psi\Big)(x')\Big|^2\,dx' \leq C\frac{1}{|t-s|^2}\|\psi\|^2_{L^2(\Sigma;\mathbb{C}^N)}.$$

This estimate, the definition of the norm in $H^1(\Sigma;\mathbb{C}^N)$, see (2.2), and (B.11) imply (B.13).

Step 4. By Proposition 2.2 (i), we have $H^{1/2}(\Sigma;\mathbb{C}^N) = [L^2(\Sigma;\mathbb{C}^N), H^1(\Sigma;\mathbb{C}^N)]_{1/2}$ and using that also $L^2(\Sigma;\mathbb{C}^N) = [L^2(\Sigma;\mathbb{C}^N), L^2(\Sigma;\mathbb{C}^N)]_{1/2}$ we conclude from the bounds (B.11) and (B.13) together with (xiii) from Section 2.1 that

$$\begin{aligned}
&\|b_{t,s,\varepsilon}(z)\|_{L^2(\Sigma;\mathbb{C}^N)\to H^{1/2}(\Sigma;\mathbb{C}^N)}\\
&\qquad\le C\|b_{t,s,\varepsilon}(z)\|^{1/2}_{L^2(\Sigma;\mathbb{C}^N)\to L^2(\Sigma;\mathbb{C}^N)}\|b_{t,s,\varepsilon}(z)\|^{1/2}_{L^2(\Sigma;\mathbb{C}^N)\to H^1(\Sigma;\mathbb{C}^N)}\\
&\qquad\le C\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}.
\end{aligned}$$

This completes the proof of Proposition B.2. □

Lemma B.3. *Let $\varepsilon \in (0,\varepsilon_{ABC})$. Then, the operator-valued function*

$$\begin{aligned}
&F : (-1,1)^2 \to \mathcal{L}(L^2(\Sigma;\mathbb{C}^N), H^{1/2}(\Sigma;\mathbb{C}^N)),\\
&F(t,s) = \begin{cases} b_{t,s,\varepsilon}(z), & \text{if } t \neq s,\\ 0, & \text{if } t = s,\end{cases}
\end{aligned}$$

is measurable.

Proof. It suffices to prove that $(F(\cdot,\cdot)\varphi,\psi)_{H^{1/2}(\Sigma;\mathbb{C}^N)}$ is measurable on $(-1,1)^2$ for all $\varphi \in L^2(\Sigma;\mathbb{C}^N)$ and $\psi \in H^{1/2}(\Sigma;\mathbb{C}^N)$; cf. Definition 2.13. For this, we prove that the function $(F(\cdot,\cdot)\varphi,\psi)_{H^{1/2}(\Sigma;\mathbb{C}^N)}$ is continuous on $\mathcal{O} := (-1,1)^2 \setminus \{(t,t) : t \in (-1,1)\}$. Let $(t,s) \in \mathcal{O}$ be fixed and let us consider the case $t > s$. We choose a sequence $((t_n,s_n))_{n\in\mathbb{N}}$ in $\mathcal{O}$ which converges to (t,s). It is no restriction to assume that $\frac{3}{2}(t_n - s_n) > t - s > \frac{1}{2}(t_n - s_n)$ holds for all $n \in \mathbb{N}$. Then,

$$L(x_\Sigma,y_\Sigma,t_n,s_n) = |x_\Sigma - y_\Sigma| + \varepsilon|t_n - s_n| < |x_\Sigma - y_\Sigma| + 2\varepsilon|t-s| \le 2L(x_\Sigma,y_\Sigma,t,s)$$

and in a similar way

$$L(x_\Sigma,y_\Sigma,t_n,s_n)^{1-\theta} \le \left(\frac{2}{3}\right)^{1-\theta} L(x_\Sigma,y_\Sigma,t,s)^{1-\theta}.$$

Moreover, as $|t_n - s_n| \ge 0 = |t-s| - |t-s| \ge |t-s| - 2$, one has

$$e^{-cL(x_\Sigma,y_\Sigma,t_n,s_n)} \le e^{-c(|x_\Sigma-y_\Sigma|+\varepsilon|t-s|)+2c\varepsilon} \le e^{2c\varepsilon_{ABC}}e^{-cL(x_\Sigma,y_\Sigma,t,s)}.$$

Combining Lemma B.1 (i) with the latter three displayed formulas yields the existence of a constant $C > 0$ which is independent of $x_\Sigma, y_\Sigma, t, s, t_n, s_n$, and ε such that

$$\begin{aligned}
|\Delta G_z(x_\Sigma,y_\Sigma,t_n,s_n)| &\le C\varepsilon L(x_\Sigma,y_\Sigma,t_n,s_n)^{1-\theta}\ e^{-cL(x_\Sigma,y_\Sigma,t_n,s_n)}\\
&\le C\varepsilon L(x_\Sigma,y_\Sigma,t,s)^{1-\theta}e^{-cL(x_\Sigma,y_\Sigma,t,s)}.
\end{aligned}\tag{B.14}$$

We claim that $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ converges weakly to $b_{t,s,\varepsilon}(z)\varphi$ in $L^2(\Sigma;\mathbb{C}^N)$. Let the function $\gamma \in L^2(\Sigma;\mathbb{C}^N)$ be fixed. Then,

$$
\begin{aligned}
&((b_{t_n,s_n,\varepsilon}(z) - b_{t,s,\varepsilon}(z))\varphi,\gamma)_{L^2(\Sigma;\mathbb{C}^N)}\\
&\quad= \int_\Sigma\int_\Sigma \langle (\Delta G_z(x_\Sigma,y_\Sigma,t_n,s_n) - \Delta G_z(x_\Sigma,y_\Sigma,t,s))\varphi(y_\Sigma),\gamma(x_\Sigma)\rangle \, d\sigma(y_\Sigma)\, d\sigma(x_\Sigma).
\end{aligned}
$$

The integrand on the right-hand side converges pointwise almost everywhere to zero, as $n\to\infty$. Moreover, (B.14) shows that the integrand is bounded by $M(x_\Sigma,y_\Sigma)|\varphi(y_\Sigma)||\gamma(x_\Sigma)|$ with

$$
M(x_\Sigma,y_\Sigma) := C\varepsilon L(x_\Sigma,y_\Sigma,t,s)^{1-\theta} e^{-cL(x_\Sigma,y_\Sigma,t,s)}.
$$

Applying the Cauchy-Schwarz inequality twice, Fubini's theorem, and the symmetry relation $M(x_\Sigma,y_\Sigma) = M(y_\Sigma,x_\Sigma)$ yields

$$
\begin{aligned}
&\left(\int_\Sigma\int_\Sigma M(x_\Sigma,y_\Sigma)|\varphi(y_\Sigma)||\gamma(x_\Sigma)|\, d\sigma(y_\Sigma)\, d\sigma(x_\Sigma)\right)^2\\
&\quad\le \int_\Sigma\left(\int_\Sigma M(x_\Sigma,y_\Sigma)|\varphi(y_\Sigma)|\, d\sigma(y_\Sigma)\right)^2 d\sigma(x_\Sigma)\|\gamma\|^2_{L^2(\Sigma;\mathbb{C}^N)}\\
&\quad\le \int_\Sigma\int_\Sigma M(x_\Sigma,y_\Sigma)|\varphi(y_\Sigma)|^2\, d\sigma(y_\Sigma)\int_\Sigma M(x_\Sigma,y_\Sigma)\, d\sigma(y_\Sigma)\, d\sigma(x_\Sigma)\|\gamma\|^2_{L^2(\Sigma;\mathbb{C}^N)}\\
&\quad\le C\left(\sup_{x_\Sigma\in\Sigma}\int_\Sigma M(x_\Sigma,y_\Sigma)\, d\sigma(y_\Sigma)\right)^2\|\varphi\|^2_{L^2(\Sigma;\mathbb{C}^N)}\|\gamma\|^2_{L^2(\Sigma;\mathbb{C}^N)}.
\end{aligned}
$$

Furthermore, with (B.10) we see that

$$
\sup_{x_\Sigma\in\Sigma}\int_\Sigma M(x_\Sigma,y_\Sigma)\, d\sigma(y_\Sigma) \le C\varepsilon(1+|\log(\varepsilon|t-s|)|) < \infty.
$$

Hence, $((b_{t_n,s_n,\varepsilon}(z) - b_{t,s,\varepsilon}(z))\varphi,\gamma)_{L^2(\Sigma;\mathbb{C}^N)} \to 0$ for $n\to\infty$ by applying the dominated convergence theorem. Since $\gamma\in L^2(\Sigma;\mathbb{C}^N)$ was arbitrary, we conclude that $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ converges weakly to $b_{t,s,\varepsilon}(z)\varphi$ in $L^2(\Sigma;\mathbb{C}^N)$.

Next, we show that $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ converges weakly to $b_{t,s,\varepsilon}(z)\varphi$ in the space $H^{1/2}(\Sigma;\mathbb{C}^N)$, which shows the claimed continuity. For this, we note that Proposition B.2 and $\frac{3}{2}(t_n - s_n) > t - s > 0$ imply that $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ is a bounded sequence in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Let us assume that $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ does not converge weakly to $b_{t,s,\varepsilon}(z)\varphi$ in $H^{1/2}(\Sigma;\mathbb{C}^N)$. Then, the $H^{1/2}$-boundedness implies that there exists a weakly convergent subsequence $(b_{t_{n_k},s_{n_k},\varepsilon}(z)\varphi)_{k\in\mathbb{N}}$ which converges to some $\varphi' \in H^{1/2}(\Sigma;\mathbb{C}^N)$ with $\varphi' \ne b_{t,s,\varepsilon}(z)\varphi$. However, in this case $(b_{t_{n_k},s_{n_k},\varepsilon}(z)\varphi)_{k\in\mathbb{N}}$ would also converge weakly to φ' in $L^2(\Sigma;\mathbb{C}^N)$ which contradicts the first part of the proof. Hence, $(b_{t_n,s_n,\varepsilon}(z)\varphi)_{n\in\mathbb{N}}$ converges weakly to $b_{t,s,\varepsilon}(z)\varphi$ in $H^{1/2}(\Sigma;\mathbb{C}^N)$ and therefore, $((b_{t_n,s_n,\varepsilon}(z)\varphi,\psi)_{H^{1/2}(\Sigma;\mathbb{C}^N)})_{n\in\mathbb{N}}$ converges to $(b_{t,s,\varepsilon}(z)\varphi,\psi)_{H^{1/2}(\Sigma;\mathbb{C}^N)}$ for all $\psi\in H^{1/2}(\Sigma;\mathbb{C}^N)$. □

After all these preliminary considerations we are prepared to prove (4.34).

Proof of (4.34). Let $f \in \mathcal{B}^0(\Sigma)$ be fixed. Using Proposition B.2, the Cauchy-Schwarz inequality, and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-1}^{1}\left(\int_{-1}^{1}\|b_{t,s,\varepsilon}(z)f(s)\|_{H^{1/2}(\Sigma;\mathbb{C}^N)}\,ds\right)^2 dt\\
&\leq C\int_{-1}^{1}\left(\int_{-1}^{1}\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}\|f(s)\|_{L^2(\Sigma;\mathbb{C}^N)}\,ds\right)^2 dt\\
&\leq C\int_{-1}^{1}\left(\int_{-1}^{1}\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}\,ds\right.\\
&\qquad\left.\cdot\int_{-1}^{1}\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}\|f(s)\|^2_{L^2(\Sigma;\mathbb{C}^N)}\,ds\right)dt\\
&\leq C\sup_{s\in(-1,1)}\left(\int_{-1}^{1}\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}\,dt\right)^2\int_{-1}^{1}\|f(s)\|^2_{L^2(\Sigma;\mathbb{C}^N)}\,ds\\
&\leq C\varepsilon(1+|\log(\varepsilon)|)\|f\|_0^2,
\end{aligned}$$

where we used that for $\varepsilon>0$ sufficiently small one has

$$\begin{aligned}
\sup_{s\in(-1,1)}&\int_{-1}^{1}\big(\varepsilon(1+|\log(\varepsilon|t-s|)|)\big)^{1/2}\frac{1}{|t-s|^{1/2}}\,dt\\
&\leq\int_{-2}^{2}\big(\varepsilon(1+|\log(\varepsilon|\tau|)|)\big)^{1/2}\frac{1}{|\tau|^{1/2}}\,d\tau\\
&\leq C\varepsilon^{1/2}(1+|\log(\varepsilon)|)^{1/2}\int_{-2}^{2}(1+|\log(|\tau|)|)^{1/2}\frac{1}{|\tau|^{1/2}}\,d\tau\\
&\leq C\varepsilon^{1/2}(1+|\log(\varepsilon)|)^{1/2}.
\end{aligned}$$

Combined with Lemma B.3, (2.8) and Proposition 2.15 this shows that the Bochner integral $\int_{-1}^{1} b_{t,s,\varepsilon}(z)f(s)\,ds \in H^{1/2}(\Sigma;\mathbb{C}^N)$ exists for a.e. $t\in(-1,1)$ and that the function $t\mapsto\int_{-1}^{1} b_{t,s,\varepsilon}(z)f(s)\,ds\in H^{1/2}(\Sigma;\mathbb{C}^N)$ is measurable. Hence, the mapping

$$\begin{aligned}
\mathbb{B}_\varepsilon(z)&:\mathcal{B}^0(\Sigma)\to\mathcal{B}^{1/2}(\Sigma),\\
\mathbb{B}_\varepsilon(z)f(t)&:=\int_{-1}^{1} b_{t,s,\varepsilon}(z)f(s)\,ds,
\end{aligned}\tag{B.15}$$

is well-defined, bounded, and $\|\mathbb{B}_\varepsilon(z)\|_{0\to1/2}\leq C\varepsilon^{1/2}(1+|\log(\varepsilon)|^{1/2})$. By (B.2), (B.3), and Proposition 2.18 (iii) we also have

$$(\widetilde{B}_\varepsilon(z)-\overline{B}_\varepsilon(z))f(t)=\int_{-1}^{1} b_{t,s,\varepsilon}(z)f(s)\,ds=\mathbb{B}_\varepsilon(z)f(t)\tag{B.16}$$

for all $f \in \mathcal{B}^{1/2}(\Sigma)$. Therefore, $\widetilde{B}_\varepsilon(z) - \overline{B}_\varepsilon(z)$ can be extended to a bounded operator from $\mathcal{B}^0(\Sigma)$ to $\mathcal{B}^{1/2}(\Sigma)$ and (4.34) is true. $\square$

Appendix C. Additional results for Section 5.1.2 and Section 8.2

In this chapter, which is based on [15], we provide results which are used in Section 5.1.2 and Section 8.2. We begin by stating a fitting version of the Schur test.

Lemma C.1. *Let k be a measurable function in $\mathbb{R}^{\theta-1} \times \mathbb{R}^{\theta-1}$ with values in $\mathbb{C}^{N\times N}$ and $\widetilde{k} \in L^1(\mathbb{R}^{\theta-1})$ such that*

$$|k(x',y')| \leq \widetilde{k}(x'-y') \quad \text{for a.e. } x',y' \in \mathbb{R}^{\theta-1}.$$

Then, k induces an integral operator $K : L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N) \to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, which is bounded by $\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}$. Moreover, if $k \in C^1(\mathbb{R}^{\theta-1} \times \mathbb{R}^{\theta-1};\mathbb{C}^{N\times N})$ and

$$\sum_{l=1}^{\theta-1} \Big| \frac{d}{dx'_l} k(x',y') \Big| \leq \widetilde{k}(x'-y') \quad \text{for a.e. } x',y' \in \mathbb{R}^{\theta-1},$$

then the induced operator K also acts as bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and the corresponding operator norm is bounded by $C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}$, where $C > 0$ does not depend on k or $\widetilde{k}$.

Proof. The first assertion is an immediate consequence of the Schur test; see for instance [44, Chapter III, Example 2.4]. Next, let us prove the second assertion. We start by choosing $g \in C_0^\infty(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. Our assumptions and dominated convergence show that in this case, Kg is differentiable and

$$\partial_l(Kg)(x') = \int_{\mathbb{R}^{\theta-1}} \frac{d}{dx'_l} k(x',y') g(y')\, dy'.$$

Hence, applying the Schur test shows

$$\|Kg\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq C\|\widetilde{k}\|_{L^1(\mathbb{R}^{\theta-1})}\|g\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}.$$

The fact that $C_0^\infty(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ is dense in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, see [54, the text above eq. (3.22)], the completeness of $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and the L^2-continuity of K imply that the estimate is also valid for $g \in L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. $\square$

Our next goal is to use the functions $\vartheta^a_{n'}$, $n' \in \mathbb{Z}$, from Corollary A.3 (for $n = \theta-1$) to construct operators based on a uniformly bounded sequence of operators. We start by providing a useful variant of the Cotlar-Stein lemma.

Lemma C.2. *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, let $(\mathcal{A}_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a family of uniformly bounded operators acting from $\mathcal{H}$ to $\mathcal{G}$. Moreover, assume that there exists a number $M \in \mathbb{N}$ such that for every $n' \in \mathbb{Z}^{\theta-1}$ exist at most M indices $m' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n'}^*\mathcal{A}_{m'}$ and $\mathcal{A}_{n'}\mathcal{A}_{m'}^*$ are nonzero operators. Then, the family $(\mathcal{A}_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is strongly summable (in the sense of* (v) *of Section 2.1) and for $\mathcal{A} = \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \mathcal{A}_{n'}$ the estimate*

$$\|\mathcal{A}\|_{\mathcal{H}\to\mathcal{G}} \leq M \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}}$$

is valid.

Proof. Our assumptions guarantee

$$\sup_{n'\in\mathbb{Z}^{\theta-1}} \sum_{m'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\mathcal{A}_{m'}^*\|_{\mathcal{G}\to\mathcal{G}}^{1/2} \leq M \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}},$$
$$\sup_{n'\in\mathbb{Z}^{\theta-1}} \sum_{m'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}^*\mathcal{A}_{m'}\|_{\mathcal{H}\to\mathcal{H}}^{1/2} \leq M \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\mathcal{A}_{n'}\|_{\mathcal{H}\to\mathcal{G}}.$$

Hence, the assertions follow from the Cotlar-Stein lemma; see [38, Lemma 18.6.5]. □

Proposition C.3. *Let $a \in (0, b)$ for a $b > 0$, $(\vartheta_{n'}^a)_{n'\in\mathbb{Z}^{\theta-1}}$ be the sequence from Corollary A.3 (for $n = \theta - 1$) and $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be a sequence of uniformly bounded operators in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. Then,*

$$A = \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a A_{n'} \vartheta_{n'}^a$$

is a well-defined operator in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ which is bounded by $11^{\theta-1} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|A_{n'}\|_{0\to0}$. Moreover, if $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ is also a family of uniformly bounded operators acting from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$, then A also acts as a bounded operator from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$ and $\|A\|_{0\to1} \leq \frac{C}{a} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|A_{n'}\|_{0\to1}$, where $C > 0$ does not depend on $a \in (0, b)$.

Proof. Let us start by proving the assertions where we consider A and $A_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, as operators acting from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^0(\mathbb{R}^{\theta-1})$. We set $\mathcal{A}_{n'} := \vartheta_{n'}^a A_{n'} \vartheta_{n'}^a$. Since a fixed ball $B(an', 3a)$ overlaps with at most $11^{\theta-1}$ balls of the type $B(am', 3a)$, $m' \in \mathbb{Z}^{\theta-1}$, there exist for every $n' \in \mathbb{Z}^{\theta-1}$ at most $M = 11^{\theta-1}$ indices $m' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n'}\mathcal{A}_{m'}^* \neq 0$ and $\mathcal{A}_{n'}^*\mathcal{A}_{m'} \neq 0$. Moreover, for all $n' \in \mathbb{Z}^{\theta-1}$ we have

$$\|\mathcal{A}_{n'}\|_{0\to0} \leq \|\vartheta_{n'}^a\|_{0\to0} \|A_{n'}\|_{0\to0} \|\vartheta_{n'}^a\|_{0\to0} \leq \|A_{n'}\|_{0\to0}.$$

Thus, by Lemma C.2 the assertions where A and $A_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, are considered as bounded operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ are true. Next, we assume that $A_{n'}$, $n' \in \mathbb{Z}^{\theta-1}$, act as uniformly bounded operators from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Using again the fact that a fixed ball $B(an', 3a)$ overlaps with at most $11^{\theta-1}$ balls of the type $B(am', 3a)$, $m' \in \mathbb{Z}^{\theta-1}$, shows that there exist for every $n' \in \mathbb{Z}^{\theta-1}$ at most $M = 11^{\theta-1}$ indices $m' \in \mathbb{Z}^{\theta-1}$ such that $\mathcal{A}_{n'}\mathcal{A}_{m'}^{0*1} \neq 0$ and $\mathcal{A}_{n'}^{0*1}\mathcal{A}_{m'} \neq 0$, where the expressions $\mathcal{A}_{n'}{}^{0*1}$ and $\mathcal{A}_{m'}{}^{0*1}$ denote the adjoint operators of $\mathcal{A}_{n'}$ and $\mathcal{A}_{m'}$, respectively, considered as operators mapping from $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ to $\mathcal{B}^1(\mathbb{R}^{\theta-1})$. Furthermore, applying Corollary A.3 gives us for all $n' \in \mathbb{Z}^{\theta-1}$

$$\begin{aligned}
\|\mathcal{A}_{n'}\|_{0\to1} &\leq \|\vartheta_{n'}^a\|_{1\to1}\|A_{n'}\|_{0\to1}\|\vartheta_{n'}^a\|_{0\to0} \\
&\leq \|\vartheta_{n'}^a\|_{1\to1}\|A_{n'}\|_{0\to1} \\
&\leq C\|\vartheta_{n'}^a\|_{W^1_\infty(\mathbb{R}^{\theta-1})}\|A_{n'}\|_{0\to1} \\
&\leq \frac{C}{a}\|A_{n'}\|_{0\to1} \\
&\leq \frac{C}{a}\|A_{n'}\|_{0\to1}.
\end{aligned}$$

Applying Lemma C.2 again concludes the proof. □

As a corollary we obtain a result for series of operators in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$.

Corollary C.4. *Let $a \in (0,b)$ for a $b>0$, $(\vartheta_{n'})_{n'\in\mathbb{Z}^{\theta-1}}$ be the sequence from Corollary A.3 (for $n = \theta-1$) and $(\underline{A_{n'}})_{n'\in\mathbb{Z}^{\theta-1}}$ be a sequence of uniformly bounded operators in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$. Then,*

$$\underline{A} = \sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \underline{A_{n'}} \vartheta_{n'}^a$$

is a well-defined operator in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ which is bounded by

$$11^{\theta-1} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|A_{n'}\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)}.$$

Moreover, if $(\underline{A_{n'}})_{n'\in\mathbb{Z}^{\theta-1}}$ is also a family of uniformly bounded operators acting from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$, then $\underline{A}$ also acts as a bounded operator from $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ to $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ and

$$\|\underline{A}\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \leq \tfrac{C}{a} \sup_{n'\in\mathbb{Z}^{\theta-1}} \|\underline{A_{n'}}\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)},$$

where $C>0$ does not depend on $a \in (0,b)$.

Proof. With the help of the operators $\mathfrak{J}$ and $\mathfrak{J}^*$ defined in (2.10) and (2.11) (for $\mathcal{O} = (-1,1)$), respectively, we are able to write

$$\sum_{n'\in\mathbb{Z}^{\theta-1}}^{\text{st.}} \vartheta_{n'}^a \underline{A_{n'}} \vartheta_{n'}^a = \mathfrak{J}^*\Big(\sum_{n'\in\mathbb{Z}^{\theta-1}} \vartheta_{n'}^a \big(\tfrac{1}{4}\mathfrak{J}\underline{A_{n'}}\mathfrak{J}^*\big)\vartheta_{n'}^a\Big)\mathfrak{J}.$$

Here, we used $\mathfrak{J}^*\mathfrak{J} = 2I$ and the fact that $\vartheta^a_{n'}$ commutes with $\mathfrak{J}$ and $\mathfrak{J}^*$, if one considers $\vartheta^a_{n'}$ once as a multiplication operator in $\mathcal{B}^0(\mathbb{R}^{\theta-1})$ (or $\mathcal{B}^1(\mathbb{R}^{\theta-1})$) and once as a multiplication operator in $L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)$ (or $H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)$). Thus, the result follows from applying Proposition C.3 to $(A_{n'})_{n'\in\mathbb{Z}^{\theta-1}} = \left(\frac{1}{2}\mathfrak{J}\underline{A_{n'}}\mathfrak{J}^*\right)_{n'\in\mathbb{Z}^{\theta-1}}$ and

$$\begin{aligned}\|\mathfrak{J}\|_{L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to 0} &= \|\mathfrak{J}^*\|_{0\to L^2(\mathbb{R}^{\theta-1};\mathbb{C}^N)} \\ &= \|\mathfrak{J}\|_{H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)\to 1} = \|\mathfrak{J}^*\|_{1\to H^1(\mathbb{R}^{\theta-1};\mathbb{C}^N)} = \sqrt{2}.\end{aligned}$$

□

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Approximation of Dirac Operators with Delta-Shell Potentials in the Norm Resolvent Sense

The present thesis is devoted to the approximation of Dirac operators with δ-shell potentials supported on the boundary of a two or three-dimensional C^2-domain. These singular potentials are used as idealized replacements for potentials which are strongly localized in a neighbourhood of the support of the δ-shell potential and they often simplify the spectral analysis. To justify the usage of such potentials it is essential to prove that Dirac operators with δ-shell potentials can be approximated by Dirac operators with strongly localized potentials in a way which transfers the spectral properties. The most important contribution of this thesis is the establishment of conditions for the convergence of Dirac operators with strongly localized potentials in the norm resolvent sense. This type of convergence implies that the spectrum of the Dirac operator with δ-shell potential can be completely characterized by the spectra of the approximating operators and vice versa. In the special case of electrostatic and Lorentz scalar δ-shell potentials an explicit convergence condition is provided. Furthermore, counterexamples which imply the sharpness of this condition are also presented.

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