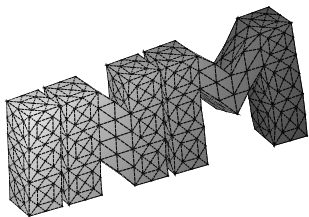# Technische Universität Graz

Coupled finite and boundary element methods for fluid–solid interaction eigenvalue problems

A. Kimeswenger, O. Steinbach, G. Unger

Berichte aus dem Institut für Numerische Mathematik

Technische Universität Graz

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Abstract

We analyze the approximation of a vibro-acoustic eigenvalue problem for an elastic body which is submerged in a compressible inviscid fluid in $\mathbb{R}^3$. As model the time-harmonic elastodynamic and the Helmholtz equation are used and are coupled in a strong sense via the standard transmission conditions on the interface between the solid and the fluid. Our approach is based on a coupling of the field equations for the solid with boundary integral equations for the fluid. The coupled formulation of the eigenvalue problem leads to a nonlinear eigenvalue problem with respect to the eigenvalue parameter since the frequency occurs nonlinearly in the used boundary integral operators for the Helmholtz equation. The nonlinear eigenvalue problem and its Galerkin discretization are analyzed within the framework of eigenvalue problems for Fredholm operator-valued functions where convergence is shown and error estimates are given. For the numerical solution of the discretized nonlinear matrix eigenvalue problem the contour integral method is a reliable method which is demonstrated by some numerical examples.

1 Introduction

In this paper we analyze a coupled finite and boundary element formulation for the numerical solution of a resonance problem arising from fluid-solid interaction in $\mathbb{R}^3$ in the time harmonic regime. We consider an elastic shell-like body in a compressible, inviscid fluid which occupies the unbounded exterior region. A strong coupling between the solid and fluid is assumed given by the equilibrium of forces and the equality of normal displacements from both media. The acoustic pressure in the fluid is modeled by the Helmholtz equation such that the formulation is also suitable for the mid-frequency range. A comprehensive theoretical analysis of the underlying resonance problem was provided in [19, Chapt. IX], however, appropriate coupled formulations for the numerical solution were not considered. Our formulation of the resonance problem is based on a field equation for the solid and

on boundary integral equations for the fluid. This formulation was first proposed in [8] for source problems and later considered in [14]. The analysis of the formulation in [8] is restricted to real frequencies and it is shown that the Fredholm alternative can be applied in the natural energy spaces for the displacement and for the acoustic pressure. By using the same arguments as in [8] we extend this result to non-real frequencies which have to be considered for resonance problems since resonance modes of the fluid-solid interaction have a damping in time and therefore the resonances are non-real. The coupled eigenvalue problem formulation is nonlinear in the eigenvalue parameter since the frequency occurs nonlinearly in the boundary integral equation for the acoustic pressure due to the nonlinear dependence of the frequency in the fundamental solution of the Helmholtz equation. The eigenvalue problem is analyzed in the framework of eigenvalue problems for holomorphic Fredholm operator-valued functions [15]. For the numerical solution of the eigenvalue problem a conforming Galerkin discretization is applied and analyzed. General results for the discretization of eigenvalue problems for holomorphic Fredholm operator-valued functions [10, 11, 22, 23, 25] are used to show convergence and to give error estimates. The discretized nonlinear algebraic eigenvalue problem can be solved with the contour integral method [2] which is demonstrated by some numerical examples.

2 Formulations of the eigenvalue problem

We consider a homogeneous and isotropic elastic shell-like body $\Omega_S \subset \mathbb{R}^3$ which is surrounded by a compressible inviscid fluid filling the exterior unbounded domain Ω_F. The domain Ω_S is assumed to be a bounded Lipschitz domain with piecewise smooth boundary. The interface between the elastic body Ω_S and the fluid domain Ω_F is denoted by Γ, whereas Γ_i denotes the interior boundary of Ω_S, see Fig. 1. We assume a time-harmonic behavior of the displacement field $\mathbf{U}$ of the elastic body and of the acoustic pressure P of the fluid of the form

$$\mathbf{U}(\mathbf{x},t) = \mathrm{Re}\left(e^{-i\omega t}\mathbf{u}(\mathbf{x})\right), \qquad P(\mathbf{x},t) = \mathrm{Re}\left(e^{-i\omega t}p(\mathbf{x})\right).$$

The Navier equations and the Helmholtz equation are used with the standard strong coupling conditions between the elastic body and the fluid for the formulation of the eigenvalue problem. In addition, a so-called outgoing radiation condition for the acoustic pressure in Ω_F is imposed. The eigenvalue problem reads then as follows: Find ω with $\mathrm{Re}(\omega) > 0$ and $\mathbf{u} \in \mathbf{H}^1(\Omega_S)$, $p \in H^1_{\mathrm{loc}}(\Omega_F)$ with $(\mathbf{u},p) \neq (\mathbf{0},0)$ such that

$$\begin{aligned}
-\varrho_S\omega^2\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = 0 \quad &\text{in } \Omega_S, \qquad T\mathbf{u} = 0 \quad \text{on } \Gamma_i, \\
-\Delta p - k^2 p = 0 \quad &\text{in } \Omega_F, \qquad p \text{ is outgoing} \\
\rho_F\omega^2\mathbf{n}\cdot\mathbf{u} = \partial_{\mathbf{n}}p \quad &\text{and} \quad T\mathbf{u} = -p\mathbf{n} \quad \text{on } \Gamma.
\end{aligned} \tag{1}$$

Here, λ and μ are the Lamé parameters, ρ_S and ρ_F are the density of the solid and the fluid, respectively, ω is the frequency parameter and $k = \frac{\omega}{c}$ is the wave number, where c is the speed of sound in the fluid. T is the boundary stress operator and $\mathbf{n}$ denotes the unit

normal vector on Γ pointing out of Ω_S. As radiation condition for the pressure p we impose that p is outgoing, i.e., p has outside of any ball $B_{r_0}(0)$ which contains Ω_S a representation of the form

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} h_n^1(kr) Y_n^m\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \quad \text{for } r = \|\mathbf{x}\| \geq r_0, \tag{2}$$

where h_n^1 are the spherical Hankel functions and Y_n^m are the spherical harmonics. This radiation condition describes outgoing waves and for positive wave-numbers this radiation condition coincides with the Sommerfeld radiation condition, see [13, Remark 2.1].

In the next theorem first spectral properties of the eigenvalue problem (1) are summarized.

Theorem 2.1. *Let $\omega \in \mathbb{C}$, $\mathrm{Re}(\omega) > 0$, and $\mathbf{u} \in \mathbf{H}^1(\Omega_S)$, $p \in H^1_{\mathrm{loc}}(\Omega_F)$ with $(\mathbf{u}, p) \neq (\mathbf{0}, 0)$ be a solution of* (1). *Then:*

a) $\mathrm{Im}(\omega) \leq 0$.

b) If $\omega \in \mathbb{R}$, then $T\mathbf{u} = 0 = \mathbf{n} \cdot \mathbf{u}$ on Γ and $p = 0$ in Ω_F. Such an ω is referred to as Jones frequency.

Proof. The assertions were proven in [16, Sect.2] for domains Ω_S with smooth boundary and for which $\mathbb{R}^3 \setminus \Omega_S$ is simply connected. Since the proof can be done in the same way if Ω_S is a Lipschitz domain and if it has homogeneous Neumann boundary conditions on Γ_i, it is here omitted. $\square$

The imaginary part of an eigenvalue of (1) describes the damping of the corresponding waves $\mathbf{U}$ and P in time, namely we have, $\mathbf{U}(\mathbf{x}, t) = e^{\mathrm{Im}(\omega)t} \mathrm{Re}\left(e^{-i\,\mathrm{Re}(\omega)t}\mathbf{u}(\mathbf{x})\right)$ and $p(\mathbf{x}, t) = e^{\mathrm{Im}(\omega)t} \mathrm{Re}\left(e^{-i\,\mathrm{Re}(\omega)t}p(\mathbf{x})\right)$. Therefore in practical applications those eigenvalues are of particular interest for which the absolute value of the imaginary part is small.
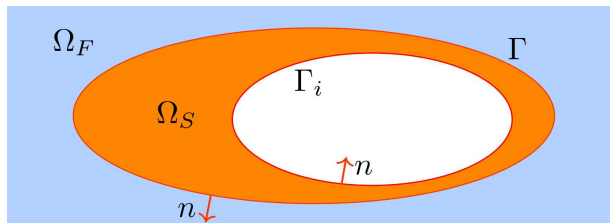

Figure 1: Computational domain

2.1 A coupled formulation of the eigenvalue problem

For the coupled formulation of the eigenvalue problem (1) we use a coupled field and boundary integral formulation which was first presented and analyzed in [8] and which was later considered in [14]. The focus in [8] were source problems and therefore the analysis was restricted to positive frequencies. By using similar arguments as in [8] we will show that the Fredholm property of the mentioned formulation is valid for all non-zero frequencies. The coupled formulation of the eigenvalue problem is a nonlinear eigenvalue problem with respect to the eigenvalue parameter ω since the wavenumber occurs nonlinearly in the boundary integral equations. However, as we will show, the dependence of the eigenvalue parameter in the coupled eigenvalue problem formulation is holomorphic.

For the representation of the acoustic pressure p in the unbounded region Ω_F we use the representation formula for outgoing solutions of the Helmholtz equation [20]

$$p(\mathbf{x}) = -\int_\Gamma U_k^*(\mathbf{x}, \mathbf{y}) \partial_\mathbf{n} p(\mathbf{y}) ds_\mathbf{y} + \int_\Gamma \partial_{\mathbf{n},\mathbf{y}} U_k^*(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) ds_\mathbf{y}, \quad \mathbf{x} \in \Omega_F, \qquad (3)$$

where

$$U_k^*(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{\|\mathbf{x} - \mathbf{y}\|}$$

is the fundamental solution. The application of the Dirichlet and Neumann trace to (3) leads to the standard boundary integral equations [17, 21]

$$p(\mathbf{x}) = -\left(V(k)\partial_\mathbf{n} p\right)(\mathbf{x}) + \sigma(\mathbf{x})u(\mathbf{x}) + (K(k)p)(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Gamma, \qquad (4)$$

$$\partial_\mathbf{n} p(\mathbf{x}) = \sigma(\mathbf{x})\partial_\mathbf{n} p(\mathbf{x}) - \left(K'(k)\partial_\mathbf{n} p\right)(\mathbf{x}) - (D(k)p)(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Gamma, \qquad (5)$$

where $\sigma(\mathbf{x}) = \frac{1}{2}$ almost everywhere on Γ, and where the single layer operator $V(k)$, the double layer operator $K(k)$, the adjoint double layer operator $K'(k)$ and the hypersingular operator $D(k)$ are formally defined in the following way:

$$(V(k)t)(\mathbf{x}) = \int_\Gamma U_k^*(\mathbf{x}, \mathbf{y}) t(\mathbf{y}) ds_\mathbf{y}, \qquad (K(k)q)(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n},\mathbf{y}} U_k^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) ds_\mathbf{y}$$

$$(K'(k)t)(\mathbf{x}) = \partial_{\mathbf{n},\mathbf{x}} \int_\Gamma U_k^*(\mathbf{x}, \mathbf{y}) t(\mathbf{y}) ds_\mathbf{y}, \quad (D(k)q)(\mathbf{x}) = \partial_{\mathbf{n},\mathbf{x}} \int_\Gamma \partial_{\mathbf{n},\mathbf{y}} U_k^*(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) ds_\mathbf{y}.$$

Note that integral representations of the boundary integral operators are only valid for sufficiently smooth t and q, see e. g., [17, Thm.7.4]. The boundary integral operators are bounded mappings for the indicated function spaces below [17, Sect. 6.2]:

$$V(k) : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \qquad K(k) : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),$$

$$K'(k) : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \qquad D(k) : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma),$$

where $s \in [-1/2, 1/2]$.

The coupled variational formulation of the eigenvalue problem (1) is derived by combining the standard variational formulation for the displacement $\mathbf{u}$ in Ω_S and boundary

4

integral equations of the pressure p on Γ together with the transmission conditions. The variational formulation of the linear elasticity part of (1) reads

$$-\rho_S\omega^2\langle\mathbf{u},\mathbf{v}\rangle_{\Omega_S} + a_S(\mathbf{u},\mathbf{v}) - \langle T\mathbf{u},\mathbf{v}\rangle_\Gamma = 0 \quad \text{for all } \mathbf{v} \in \mathbf{H}^1(\Omega_S) \tag{6}$$

with

$$\langle\mathbf{u},\mathbf{v}\rangle_{\Omega_S} := \int_{\Omega_S} \mathbf{u}\cdot\overline{\mathbf{v}}\,d\mathbf{x}, \qquad \langle T\mathbf{u},\mathbf{v}\rangle_\Gamma := \int_\Gamma T\mathbf{u}\cdot\overline{\mathbf{v}}\,ds,$$

$$a_S(\mathbf{u},\mathbf{v}) := \lambda\int_{\Omega_S}\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{v}}\,d\mathbf{x} + \frac{\mu}{2}\int_{\Omega_S}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right):\left(\nabla\overline{\mathbf{v}}+(\nabla\overline{\mathbf{v}})^\top\right)\,d\mathbf{x}.$$

The latter term of the left hand side of (6) can be written as

$$\begin{aligned}
\langle Tu,\mathbf{v}\rangle_\Gamma &= \langle-\mathbf{n}p,\mathbf{v}\rangle_\Gamma = \langle\mathbf{n}[V(k)\partial_\mathbf{n}p - (\tfrac{1}{2}I + K(k))p],\mathbf{v}\rangle_\Gamma \\
&= \langle\mathbf{n}[\rho_F\omega^2 V(k)\mathbf{n}\cdot\mathbf{u} - (\tfrac{1}{2}I + K(k))p],\mathbf{v}\rangle_\Gamma,
\end{aligned} \tag{7}$$

where we have used the transmission conditions and the boundary integral equation (4). Next, we take the boundary integral equation (5) for the pressure p, plug in the transmission condition $\rho_F\omega^2\mathbf{n}\cdot\mathbf{u} = \partial_\mathbf{n}p$ and divide by $\rho_F\omega^2$ which yields

$$\langle(\tfrac{1}{2}I + K'(k))\mathbf{n}\cdot\mathbf{u},q\rangle_\Gamma + \frac{1}{\rho_F\omega^2}\langle D(k)p,q\rangle_\Gamma = 0 \quad \text{for all } q \in H^{1/2}(\Gamma). \tag{8}$$

Combining equations (6)-(8) yields the coupled eigenvalue problem which reads as follows: Find $\omega\in\mathbb{C}$, $\operatorname{Re}(\omega) > 0$, and $(\mathbf{u},p)\in\mathbf{H}^1(\Omega_S)\times H^{1/2}(\Gamma)$, $(\mathbf{u},p)\neq(\mathbf{0},0)$, such that

$$\begin{aligned}
-\rho_S\omega^2\langle\mathbf{u},\mathbf{v}\rangle_{\Omega_S} + a_S(\mathbf{u},\mathbf{v}) - \langle\mathbf{n}[\rho_F\omega^2 V(k)\mathbf{n}\cdot\mathbf{u}],\mathbf{v}\rangle_\Gamma + \langle\mathbf{n}[(\tfrac{1}{2}I + K(k))p],\mathbf{v}\rangle_\Gamma \\
+ \langle(\tfrac{1}{2}I + K'(k))\mathbf{n}\cdot\mathbf{u},q\rangle_\Gamma + \frac{1}{\rho_F\omega^2}\langle D(k)p,q\rangle_\Gamma = 0 \quad (9)
\end{aligned}$$

is satisfied for all $(\mathbf{v},q)\in\mathbf{H}^1(\Omega_S)\times H^{1/2}(\Gamma)$. Note, that the eigenvalue parameter ω occurs nonlinearly in the eigenvalue problem formulation (9) due to the wavenumber representation $k = \frac{\omega}{c}$. In the following we will consider the operator notation of the eigenvalue problem (9):

$$\mathcal{A}(\omega)\begin{pmatrix}\mathbf{u}\\p\end{pmatrix} := \begin{pmatrix}-\omega^2\rho_S M_s + A_S - N^*\rho_F\omega^2 V(k)N & N^*(\tfrac{1}{2}I + K(k))\\ (\tfrac{1}{2}I + K'(k))N & \frac{1}{\rho_F\omega^2}D(k)\end{pmatrix}\begin{pmatrix}\mathbf{u}\\p\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\0\end{pmatrix}, \tag{10}$$

where $M_S, A_S : \mathbf{H}^1(\Omega_S) \to \mathbf{H}^1(\Omega_S)^*$ are the operators related to the sesquilinear forms $\langle\cdot,\cdot\rangle_{\Omega_S}$ and $a_S(\cdot,\cdot)$, and where $N : \mathbf{H}^1(\Omega_S) \to H^{-1/2}(\Gamma)$ is defined by

$$N\mathbf{u} = \mathbf{n}\cdot\mathbf{u}_{|\Gamma} \qquad \text{for } \mathbf{u} \in \mathbf{H}^1(\Omega_S).$$

5

An eigenvalue problem of the form (10) is referred to as an eigenvalue problem for the operator-valued function $\mathcal{A}$.

In the rest of this subsection we will derive the essential properties of the operator-valued function $\mathcal{A}$ which are needed for the spectral analysis of the eigenvalue problem (10). First we will show that for $\omega \in \mathbb{C} \setminus \{0\}$ the operator $\mathcal{A}(\omega)$ defines a Fredholm operator from $\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma) =: \mathcal{H}$ into $\mathbf{H}^1(\Omega_S)^* \times H^{-1/2}(\Gamma) = \mathcal{H}^*$ where $\mathcal{H}$ is endowed with the natural graph norm. For positive ω this was already shown in [8, Thm. 4.2].

Theorem 2.2. *Let $\omega \in \mathbb{C} \setminus \{0\}$. Then, $\mathcal{A}(\omega) : \mathcal{H} \to \mathcal{H}^*$ is a compact perturbation of the operator*

$$\mathcal{B}(\omega) := \begin{pmatrix} A_s + M_s & 0 \\ 0 & \frac{1}{\rho_F \omega^2} \widetilde{D}(0) \end{pmatrix}$$

where $\widetilde{D}(0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the modified hypersingular boundary integral operator of the Laplace equation [21] defined as

$$\langle \widetilde{D}(0)u, v \rangle_\Gamma := \langle D(0)u, v \rangle_\Gamma + \langle u, 1 \rangle_\Gamma \langle v, 1 \rangle_\Gamma \quad \text{for all } u, v \in H^{1/2}(\Gamma).$$

Further, the operator $\mathcal{B}(\omega)^{-1} : \mathcal{H}^* \to \mathcal{H}$ exists and is linear and bounded.
If $|\operatorname{Re}(\omega)| > |\operatorname{Im}(\omega)|$, then $\mathcal{B}(\omega)$ is elliptic, i. e., there exists a constant $\alpha(\omega) > 0$ such that

$$\operatorname{Re} \langle \mathcal{B}(\omega)z, \overline{z} \rangle_{\mathcal{H}^* \times \mathcal{H}} \geq \alpha(\omega) \|z\|^2 \quad \text{for all } z \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}^* \times \mathcal{H}}$ denotes the duality pairing of $\mathcal{H}$ and $\mathcal{H}^*$.

Proof. We can write

$$\mathcal{A}(\omega) = \mathcal{B}(\omega) + \underbrace{\begin{pmatrix} (-1 - \omega^2 \rho_s)M_S - N^* \rho_F \omega^2 V(k)N & N^*(\frac{1}{2}I + K(k)) \\ \omega^2 \rho_F(\frac{1}{2}I + K(k))N & D(k) - \widetilde{D}(0) \end{pmatrix}}_{=: \mathcal{C}(\omega)}. \tag{11}$$

The invertibility of $\mathcal{B}(\omega) : \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma) \to \mathbf{H}^1(\Omega_S)^* \times H^{-1/2}(\Gamma)$ follows from the ellipticity of $A_s + M_s : \mathbf{H}^1(\Omega_S) \to \mathbf{H}^1(\Omega_S)^*$ and of $\widetilde{D}(0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ [7, 21].

Let $a := \operatorname{Re}(\omega)$ and $b := \operatorname{Im}(\omega)$. If $|a| > |b|$, then for $z = (\mathbf{u}, p) \in \mathcal{H}$ we have

$$\operatorname{Re} \langle \mathcal{B}(\omega)z, \overline{z} \rangle = \langle (A_S + M_S)\mathbf{u}, \mathbf{u} \rangle_\Omega + \frac{a^2 - b^2}{\rho_F d} \langle \widetilde{D}(0)p, p \rangle_\Gamma \geq \alpha(\omega) \|z\|^2_{\mathcal{H}},$$

where $d = (a^2 - b^2)^2 + (2ab)^2$.

The operator $\mathcal{C}(\omega) : \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma) \to \mathbf{H}^1(\Omega_S)^* \times H^{-1/2}(\Gamma)$ is compact since each block of $\mathcal{C}(\omega)$ is compact: It is well known that $M_S : H_1(\Omega) \to H_1(\Omega)^*$ and $D(k) - D(0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ [7] are compact, where from the latter follows that $D(k) - \widetilde{D}(0)$ is compact. The off-diagonal blocks of $\mathcal{C}$ are compact since $N : \mathbf{H}^1(\Omega_S) \to H^{-1/2}(\Gamma)$ is compact because the mapping $u \mapsto u_{|\Gamma} \cdot \mathbf{n}$ is bounded and linear from $\mathbf{H}^1(\Omega_S)$ into $L_2(\Gamma)$ and since $L_2(\Gamma)$ is compactly embedded in $H^{-1/2}(\Gamma)$. $\square$

6

Next we show that $\mathcal{A}$ defines a holomorphic operator-valued function.

Theorem 2.3. *The operator-valued function*

$$\mathcal{A} : \mathbb{C} \setminus \{0\} \to \mathcal{L}(\mathcal{H}, \mathcal{H}^*),$$
$$\omega \mapsto \mathcal{A}(\omega)$$

is holomorphic.

Proof. It suffices to show that the map

$$\omega \mapsto \left\langle A(\omega) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right\rangle_{\mathcal{H}^* \times \mathcal{H}}$$

is holomorphic for all $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$ for a dense subspace of $\mathcal{H} = \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$, see Theorem III.3.12 in Kato [12] and the remark following it. Let us choose $\mathbf{C}^\infty(\Omega) \cap \mathbf{H}^1(\Omega_S) \times C^\infty(\Gamma)$ as dense subspace of $\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$, then we can use integral representations of the duality product and of the boundary integral operators [17, Chapter 7, Thm. 7.4, Thm. 9.15] and the holomorphy of the mapping $\omega \mapsto \mathcal{A}(\omega)$ follows from the holomorphy of the fundamental solution $\frac{e^{i\frac{\omega}{c}\|\mathbf{x}-\mathbf{y}\|}}{\|\mathbf{x}-\mathbf{y}\|}$. $\qquad\square$

2.2 Notations and properties of eigenvalue problems for holomorphic Fredholm operator-valued functions

In this subsection we introduce notions and properties of eigenvalue problems for holomorphic Fredholm operator-valued functions where we follow [15, Appendix]. Let X, Y be reflexive Hilbert spaces and let $\Lambda \subset \mathbb{C}$ be open and connected. We assume that $A : \Lambda \to \mathcal{L}(X, Y)$ is a holomorphic operator-valued function and that $A(\lambda) : X \to Y$ is Fredholm with index zero for all $\lambda \in \Lambda$. The set

$$\rho(A) := \{\lambda \in \Lambda : \exists A(\lambda)^{-1} \in \mathcal{L}(Y, X)\}$$

is called the resolvent set of A. In the following we will assume that the resolvent set of A is not empty. The complement of the resolvent set $\rho(A)$ in Λ is called spectrum $\sigma(A)$. A number $\lambda_0 \in \Lambda$ is an eigenvalue of A if there exists a non-trivial $x_0 \in X \setminus \{0\}$ such that

$$A(\lambda_0)x_0 = 0.$$

x_0 is called an eigenelement of A corresponding to the eigenvalue λ_0. The spectrum $\sigma(A)$ has no cluster points in Λ [6, Corollary IV.8.4] and each $\lambda \in \sigma(A)$ is an eigenvalue of A which follows from the Fredholm alternative. The dimension of the nullspace $\ker A(\lambda_0)$ of an eigenvalue λ_0 is called the geometric multiplicity. An ordered collection of elements $x_0, x_1, \ldots, x_{m-1}$ in X is called a Jordan chain of λ_0 if x_0 is an eigenelement corresponding to λ_0 and if

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0) x_{n-j} = 0 \quad \text{for all } n = 0, 1, \ldots, m-1 \tag{12}$$

is satisfied, where $A^{(j)}$ denotes the j–th derivative. The length of any Jordan chain of an eigenvalue is finite [15, Lemma A.8.3]. The maximal length of a Jordan chain of the eigenvalue λ_0 is denoted by $\varkappa(A, \lambda_0)$. Elements of any Jordan chain of an eigenvalue λ_0 are called generalized eigenelements of λ_0. The closed linear hull of all generalized eigenelements of an eigenvalue λ_0 is called generalized eigenspace of λ_0 and is denoted by $G(A, \lambda_0)$. The dimension of the generalized eigenspace $G(A, \lambda_0)$ is finite [15, Prop. A.8.4] and it is referred to as algebraic multiplicity of λ_0.

2.3 Spectral properties of the coupled formulation of the eigenvalue problem

In Section 2.1 we have seen that every eigenvalue of the eigenvalue problem (1) is an eigenvalue of the eigenvalue problem for the operator-valued function $\mathcal{A}$. Vice versa, not every eigenvalue of the eigenvalue problem for $\mathcal{A}$ is an eigenvalue of (1). If k is an eigenvalue of the Neumann eigenvalue problem of the Laplacian in the bounded domain $\mathbb{R}^3 \setminus \overline{\Omega}_F$, then $\ker D(k) = \ker\left(\frac{1}{2}I + K(k)\right) \neq \{0\}$ [5, Prop. 1.2] and $(\mathbf{0}, p)$, $p \in \ker D(k) \setminus \{0\}$, is obviously an eigenelement of the eigenvalue problem for $\mathcal{A}$. However, for this eigenelement of $\mathcal{A}$ the transmission condition $T\mathbf{u} = \mathbf{n} \cdot p$ of the eigenvalue problem (1) is not fulfilled.

In the next theorem we show that if $\ker D(k) = \{0\}$, then an eigenvalue of the eigenvalue problem for $\mathcal{A}$ is also an eigenvalue of (1).

Theorem 2.4. *Let $\mathrm{Re}(\omega) > 0$ and $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$. Assume that*

$$A(\omega) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}. \tag{13}$$

If $\ker(D(k)) = \{0\}$, then $(\omega, \mathbf{u}, \tilde{p})$ is a solution of the eigenvalue problem (1), where $\tilde{p}$ is defined by

$$\tilde{p}(\mathbf{x}) = -\int_\Gamma U_k^*(\mathbf{x}, \mathbf{y}) \rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}(\mathbf{y}) ds_{\mathbf{y}} + \int_\Gamma \partial_{\mathbf{n}, \mathbf{y}} U_k^*(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega_F, \tag{14}$$

and it holds $\tilde{p} = p$ on Γ.

Proof. Let $\mathrm{Re}(\omega) > 0$ and assume that $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$ is a solution of (13). Further, let $\tilde{p}$ be defined by (14). The assumption $\ker(D(k)) = \{0\}$ implies that we can construct a unique outgoing solution $\hat{p} \in H^1_{\mathrm{loc}}(\Omega_F)$ of the Neumann problem of the Helmholtz equation

$$-\Delta\hat{p} - k^2\hat{p} = 0 \quad \text{in } \Omega_F, \qquad \partial_{\mathbf{n}}\hat{p} = \rho_F \omega^2 \mathbf{n} \cdot \mathbf{u} \quad \text{on } \Gamma, \tag{15}$$

by using the boundary integral equation (5) in the form

$$D(k)\hat{p} = -(\frac{1}{2}I + K'(k))\rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}, \tag{16}$$

for the determining $\hat{p}$ on Γ, see [20]. Then

$$\hat{p}(\mathbf{x}) = -\int_{\Gamma} U_k^*(\mathbf{x}, \mathbf{y})\rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}(\mathbf{y}) ds_{\mathbf{y}} + \int_{\Gamma} \partial_{\mathbf{n}, \mathbf{y}} U_k^*(\mathbf{x}, \mathbf{y})\hat{p}(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega_F,$$

is the unique outgoing solution of (15). The second equation of (13) gives

$$D(k)p = -(\frac{1}{2}I + K'(k))\rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}, \tag{17}$$

hence $D(k)p = D(k)\hat{p}$. Because $D(k)$ is assumed to be injective the Fredholm property of $D(k)$, see e. g. [17, Thm. 7.8], implies that $p = \hat{p}$ on Γ. Hence $\tilde{p} = \hat{p}$ in Ω_F and $p = \tilde{p}$ on Γ. Combining the boundary integral equation (5) for $\tilde{p}$ and equation (17) with $p = \tilde{p}$ we see that the transmission condition $\partial_{\mathbf{n}}\tilde{p} = \rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}$ on Γ is fulfilled.

The first equation of (13) implies that $\mathbf{u}$ solves the problem

$$-\varrho_S \omega^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_S$$

$$T\mathbf{u} = 0 \quad \text{on } \Gamma_i, \qquad T\mathbf{u} = \mathbf{n}\left(\rho_F \omega^2 V(k)\mathbf{n} \cdot \mathbf{u} - \left(\frac{1}{2}I + K(k)\right)p\right) \quad \text{on } \Gamma.$$

Since $\partial_{\mathbf{n}}\tilde{p} = \rho_F \omega^2 \mathbf{n} \cdot \mathbf{u}$ it follows from the boundary integral equation (4) and from $p = \tilde{p}$ on Γ that $-p = V(k)\rho_F \omega^2 \mathbf{n} \cdot \mathbf{u} - \left(\frac{1}{2}I + K(k)\right)p$. This shows that the transmission condition $T\mathbf{u} = -p\mathbf{n}$ on Γ is fulfilled. $\square$

Corollary 2.5. *Let* $\operatorname{Re}(\omega) > 0$ *and* $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$, $(\mathbf{u}, p) \neq (\mathbf{0}, 0)$. *Assume that*

$$A(\omega)\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}. \tag{18}$$

Then:

a) $\operatorname{Im}(\omega) \leq 0$.

b) *If* $\operatorname{Im}(\omega) = 0$, *i. e.* $\omega > 0$, *then one of the following assertions holds:*

   i) ω *is a Jones frequency, i. e.,* $T\mathbf{u} = 0$ *and* $\mathbf{n} \cdot \mathbf{u} = 0$ *on* Γ *and* $p = 0$ *in* Ω_F.

   ii) $\ker(D(k)) \neq \{0\}$

c) *If* $\operatorname{Im}(\omega) < 0$, *then* ω *is an eigenvalue of* (1) *if* $\ker(D(k)) = \{0\}$.

Proof. Assertion c) has already been shown in Thm. 2.4. For $\operatorname{Im}(\omega) > 0$ the operator $D(k)$ is elliptic, see [1]. Thm. 2.1 shows that (1) has no eigenvalue ω with $\operatorname{Im}(\omega) > 0$ therefore assertion a) is a direct consequence of Thm. 2.4. Assertion b) follows also from Thm. 2.1 and Thm. 2.4. $\square$

Remark 2.6. *In practical computations the eigenvalues for the operator-valued function D can be computed simultaneously when the eigenvalues of $\mathcal{A}$ are computed since D is the lower right block of $\mathcal{A}$. In the case that $\omega \in \sigma(\mathcal{A})$ and $\frac{\omega}{c} \in \sigma(D)$, one has to check if the transmission conditions of (1) are fulfilled in order to decide if ω is an eigenvalue of (1).*

For our numerical analysis of the approximation of the eigenvalues of $\mathcal{A}$ the consideration of the eigenvalue problem for the adjoint operator-valued function of $\mathcal{A}$ is necessary which is defined by

$$\mathcal{A}^*(\omega) := (\mathcal{A}(\overline{\omega}))^*.$$

The adjoint of the operator $\mathcal{A}(\omega)$ is defined with respect to the sesquilinear form, i. e.,

$$\left\langle \mathcal{A}(\omega) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \overline{\begin{pmatrix} \mathbf{v} \\ q \end{pmatrix}} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = \left\langle \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \overline{(\mathcal{A}(\omega))^* \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix}} \right\rangle_{\mathcal{H} \times \mathcal{H}^*}.$$

Lemma 2.7. *For $\omega \in \mathbb{C} \setminus \{0\}$ it holds*

$$(\mathcal{A}(\omega))^* = \mathcal{A}(-\overline{\omega}), \quad i.\ e.\ \mathcal{A}^*(\omega) = \mathcal{A}(-\omega).$$

Proof. It suffices to show that $\langle \mathcal{A}(\omega)(\mathbf{u}, p), (\mathbf{v}, q) \rangle_{\mathcal{H}^* \times \mathcal{H}} = \langle (\mathbf{u}, p), \mathcal{A}(-\overline{\omega})(\mathbf{v}, q) \rangle_{\mathcal{H} \times \mathcal{H}^*}$ holds for all elements $(\mathbf{u}, p), (\mathbf{v}, q)$ of a dense subspace of $\mathcal{H} = \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$. Assume that $(\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{C}^\infty(\Omega) \cap \mathbf{H}^1(\Omega_S) \times C^\infty(\Gamma)$. By definition of $\mathcal{A}$ we have

$$\left\langle \mathcal{A}(\omega) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \overline{\begin{pmatrix} \mathbf{v} \\ q \end{pmatrix}} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = \langle (-\omega^2 \rho_S M_s + A_S)\mathbf{u}, \mathbf{v} \rangle_\Omega - \langle N^* V(k) N \mathbf{u}, \mathbf{v} \rangle_\Omega$$
$$+ \langle N^*(\tfrac{1}{2}I + K(k))p, \mathbf{v} \rangle_\Omega + \langle (\tfrac{1}{2}I + K'(k))N)\mathbf{u}, q \rangle_\Gamma + \langle \frac{1}{\rho_F \omega^2} D(k)p, q \rangle_\Gamma. \quad (19)$$

Since M_S and A_S are self-adjoint with respect to $\langle \cdot, \overline{\cdot} \rangle_\Omega$ and ρ_S is real, we obtain from $-\omega^2 = \overline{-(-\overline{\omega})^2}$ for the first sesquilinear form of (19)

$$\langle (-\omega^2 \rho_S M_s + A_S)\mathbf{u}, \mathbf{v} \rangle_\Omega = \langle \mathbf{u}, (-(-\overline{\omega})^2 \rho_S M_s + A_S)\mathbf{v} \rangle_\Omega.$$

For the remaining terms of the right hand side of (19) we get with $\overline{e^{ik}} = e^{-i\overline{k}}$ by using integral representations of the sesquilinear forms and of the boundary integral operators

$$\langle N^* \rho_F \omega^2 V(k) N \mathbf{u}, \mathbf{v} \rangle_{\Omega_S} = \langle \rho_F \omega^2 V(k) N \mathbf{u}, N \mathbf{v} \rangle_\Gamma = \langle N \mathbf{u}, \rho_F (-\overline{\omega})^2 V(-\overline{k}) N \mathbf{v} \rangle_\Gamma$$
$$= \langle \mathbf{u}, N^* \rho_F (-\overline{\omega})^2 V(-\overline{k}) N \mathbf{v} \rangle_{\Omega_S}$$
$$\langle N^*(\tfrac{1}{2}I + K(k))p, \mathbf{v} \rangle_{\Omega_S} = \langle (\tfrac{1}{2}I + K(k))p, N \mathbf{v} \rangle_\Gamma = \langle p, (\tfrac{1}{2}I + K'(-\overline{k})) N \mathbf{v} \rangle_\Gamma$$
$$\langle (\tfrac{1}{2}I + K'(k))N)\mathbf{u}, q \rangle_\Gamma = \langle N \mathbf{u}, (\tfrac{1}{2}I + K^{(} - \overline{k})) q \rangle_\Gamma = \langle \mathbf{u}, N^*(\tfrac{1}{2}I + K(-\overline{k})) q \rangle_{\Omega_S}$$
$$\langle \frac{1}{\rho_F \omega^2} D(k)p, q \rangle_\Gamma = \langle p, \frac{1}{\rho_F (-\overline{\omega})^2} D(-\overline{k}) q \rangle_\Gamma,$$

from which the assertion follows. $\square$

Since $\mathcal{A}(\omega)$ is a a Fredholm operator with index 0 we conclude from $(\mathcal{A}(\omega))^* = \mathcal{A}(-\overline{\omega})$ that if $\omega \in \sigma(\mathcal{A})$, then also $-\overline{\omega} \in \sigma(\mathcal{A})$.

3 Galerkin approximation and convergence analysis

The variational formulation of the eigenvalue problem for the operator-valued function $\mathcal{A}$ reads: Find $\omega \in \mathbb{C}$ with $\mathrm{Re}(\omega) > 0$ and $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma) \neq (\mathbf{0}, 0)$ such that

$$\left\langle \mathcal{A}(\omega) \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \overline{\begin{pmatrix} \mathbf{v} \\ q \end{pmatrix}} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = 0 \tag{20}$$

is satisfied for all $(\mathbf{v}, q) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$. For the approximation of (20) we consider a piecewise linear finite element space $\mathbf{S}_h(\Omega_S)$ which are defined with respect to some admissible triangulation of Ω_S and matching piecewise linear boundary element space $S_h(\Gamma)$ on Γ. The Galerkin approximation of the eigenvalue problem (20) reads then as follows: find $\omega_h \in \mathbb{C}$ and $(\mathbf{u}_h, p_h) \in S_h(\Omega_S) \times S_h(\Gamma)$, $(\mathbf{u}_h, p_h) \neq (\mathbf{0}, 0)$, such that

$$\left\langle \mathcal{A}(\omega_h) \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix}, \overline{\begin{pmatrix} \mathbf{v}_h \\ q_h \end{pmatrix}} \right\rangle_{\mathcal{H}^* \times \mathcal{H}} = 0 \tag{21}$$

is satisfied for all $(\mathbf{v}_h, q_h) \in \mathbf{S}_h(\Omega_S) \times S_h(\Gamma)$.

For the numerical analysis of the Galerkin variational eigenvalue problem (21) we rewrite the variational formulation (20) in terms of the inner product in $\mathcal{H}$ and use an equivalent operator formulation of the Galerkin eigenvalue problem (21). This will allow us to apply directly the general results of the convergence theory for the Galerkin approximation of eigenvalue problems for holomorphic Fredholm operator-valued functions [11, 25]. Let $\mathcal{J} : \mathcal{H} \to \mathcal{H}^*$ be the Riesz map, then the operator $\mathcal{I} : \mathcal{H} \to \mathcal{H}^*$ defined by $\mathcal{I} \cdot = \overline{\mathcal{J} \cdot}$ is a linear isometry. So we can identify the anti-duality product $\langle \cdot, \overline{\cdot} \rangle_{\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)^* \times \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)}$ with the scalar product $(\cdot, \cdot)_{\mathcal{H}}$, i. e.,

$$\langle \mathcal{A}(\omega) z, \overline{w} \rangle_{\mathcal{H}^* \times \mathcal{H}} = (\mathcal{I}^* \mathcal{A}(\omega) z, w)_{\mathcal{H}}$$

holds for all $z, w \in \mathcal{H}$. Since $\mathcal{I}^* : \mathcal{H}^* \to \mathcal{H}$ is an isomorphism, the operator-valued functions $\mathcal{A}$ and $\mathcal{I}^* \mathcal{A}$ have the same eigenvalues and eigenspaces. Let $\mathcal{P}_h$ be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_h := \mathbf{S}_h(\Omega_S) \times S_h(\Gamma)$. Because of the orthogonality property of $\mathcal{P}_h$ we have $(\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega_h) z_h - \mathcal{I}^* \mathcal{A}(\omega_h) z_h, w_h)_{\mathcal{H}} = 0$ for all $z_h, w_h \in \mathcal{H}_h$, from which follows that (ω_h, z_h) is an eigenpair of (21) if and only if it is an eigenpair of the eigenvalue problem

$$\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega_h) z_h = 0. \tag{22}$$

The eigenvalue problem (22) can be interpreted as an operator representation of the Galerkin eigenvalue problem (21).

Let us summarize the properties of $\mathcal{I}^* \mathcal{A}(\omega)$ and its approximations $\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega)$ which are required in order that we can apply the convergence results and error estimates from [11, 25] to the eigenvalue problem (22), see [11, p. 390] and [25, p. 30]:

P1) $\rho(\mathcal{I}^* \mathcal{A}) \neq \emptyset$,

11

P2) $\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega) : \mathcal{H}_h \to \mathcal{H}_h$ are Fredholm operators with index zero for every $\omega \in \mathbb{C} \setminus \{0\}$,

P3) for every fixed compact $\Lambda_0 \subset \mathbb{C} \setminus \{0\}$ there exist a $C > 0$ and an $h_0 > 0$ such that $\|\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega)\| \leq C$ for all $\omega \in \Lambda_0$ and $h \in (0, h_0]$,

P4) for all $\omega \in \mathbb{C} \setminus \{0\}$ and $z \in \mathcal{H}$ we have

$$\|\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega) \mathcal{P}_h z - \mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega) z\| \to 0 \quad \text{as } h \to 0,$$

P5) the sequence $(\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega))$ converges for all $\omega \in \mathbb{C} \setminus \{0\}$ regularly to $\mathcal{I}^* \mathcal{A}(\omega)$ as $h \to 0$ for all $\omega \in \mathbb{C} \setminus \{0\}$, i. e., if $(z_{\tilde{h}})$ is a subsequence of a bounded sequence (z_h), $z_h \in \mathcal{H}_h$, such that $\mathcal{P}_{\tilde{h}} \mathcal{I}^* \mathcal{A}(\omega) z_{\tilde{h}} \to \mathcal{P}_{\tilde{h}} z$ as $\tilde{h} \to 0$ for some $z \in \mathcal{H}$, then there exists a subsequence $(z_{h'})$ of $(z_{\tilde{h}})$ and a $z' \in \mathcal{H}$ such that $\mathcal{P}_{h'} z_{h'} \to z'$ as $h' \to 0$.

The properties P1)- P4) are obviously fulfilled. The regular convergence of the sequence $(\mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega))$ to $\mathcal{I}^* \mathcal{A}(\omega)$ as $h \to 0$ follows from the decomposition $\mathcal{A}(\omega) = \mathcal{B}(\omega) + \mathcal{C}(\omega)$ as given in (11) with $R(\mathcal{I}^* \mathcal{B}(\omega)) = \mathcal{H}$ and from the fact that $\mathcal{P}_h \mathcal{I}^* \mathcal{B}(\omega)$ converges stably to $\mathcal{I}^* \mathcal{B}(\omega)$ and $\mathcal{P}_h \mathcal{I}^* \mathcal{C}(\omega)$ converges compactly to $\mathcal{I}^* \mathcal{C}(\omega)$, see [24, Prop. 5]. Here the convergence has to be understood as pointwise convergence which is fulfilled because $\mathcal{P}_h$ converges pointwise to the identity $I_{\mathcal{H}}$ due to the approximation properties of $\mathbf{S}_h(\Omega_S)$ [3, Sect. 4.4] and of $S_h(\Gamma)$ [18, Thm. 2.3]. The convergence $\mathcal{P}_h \mathcal{I}^* \mathcal{B}(\omega)$ to $\mathcal{I}^* \mathcal{B}(\omega)$ is stable since $(\mathcal{P}_h \mathcal{I}^* \mathcal{B}(\omega))^{-1} : \mathcal{H}_h \to \mathcal{H}_h$ exists and is uniformly bounded for sufficiently small $h > 0$. The compact convergence of $\mathcal{P}_h \mathcal{I}^* \mathcal{C}(\omega)$ to $\mathcal{I}^* \mathcal{C}(\omega)$ is obvious since $\mathcal{C}(\omega)$ is compact.

The approximation quality of an eigenvalue ω_h and of the corresponding eigenfunctions $(\mathbf{u}_h, p_h)$ of the Galerkin eigenvalue problem (21) depends on the approximation property of the ansatz space $\mathbf{S}_h(\Omega) \times S_h(\Gamma)$ with respect to the generalized eigenspaces $G(\mathcal{I}^* \mathcal{A}, \omega)$ and $G((\mathcal{I}^* \mathcal{A})^*, \omega)$ of the eigenvalue $\omega \in \sigma(\mathcal{I}^* \mathcal{A})$ which is approximated. We want to recall that $\sigma(\mathcal{A}) = \sigma(\mathcal{I}^* \mathcal{A})$, $G(\mathcal{A}, \omega) = G(\mathcal{I}^* \mathcal{A}, \omega)$ and $G(\mathcal{A}^*, \omega) = G((\mathcal{I}^* \mathcal{A})^*, \omega)$. For $\omega \in \sigma(\mathcal{A})$ we define

$$\delta_h(\omega) := \max_{\substack{(\mathbf{v}, q) \in G(\mathcal{A}, \omega), \\ \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega_S)} + \|q\|^2_{H^{1/2}(\Gamma)} = 1}} \text{dist}((\mathbf{v}, q), \mathbf{S}_h(\Omega_S) \times S_h(\Gamma)),$$

$$\delta_h^*(\omega) := \max_{\substack{(\mathbf{v}, q) \in G(\mathcal{A}^*, \omega), \\ \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega_S)} + \|q\|^2_{H^{1/2}(\Gamma)} = 1}} \text{dist}((\mathbf{v}, q), \mathbf{S}_h(\Omega_S) \times S_h(\Gamma)),$$

where

$$\text{dist}((\mathbf{v}, q), \mathbf{S}_h(\Omega) \times S_h(\Gamma)) := \inf_{(\mathbf{v}_h, q_h) \in \mathbf{S}_h(\Omega_S) \times S_h(\Gamma)} \sqrt{\|\mathbf{v} - \mathbf{v}_h\|^2_{\mathbf{H}^1(\Omega_S)} + \|q - q_h\|^2_{H^{1/2}(\Gamma)}}$$

for $(\mathbf{v}, q) \in \mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)$.

Theorem 3.1. *Let $\omega \in \sigma(\mathcal{A})$. Then:*

a) *There exists a sequence (ω_h) of the eigenvalues of eigenvalue problem (22) such that $\omega_h \to \omega$ as $h \to 0$.*

b) *Let $\Lambda \subset \mathbb{C} \setminus \{0\}$ be compact set such that the boundary $\partial\Lambda \subset \rho(\mathcal{A})$ and $\Lambda \cap \sigma(\mathcal{A}) = \{\omega\}$. Then there exists a $h_0 > 0$ and a constant $c > 0$ such that for all $0 < h \leq h_0$ we have*

$$|\omega_h - \omega| \leq c \left(\delta_h(\omega) \delta_h^*(\omega) \right)^{1/\varkappa} \qquad \text{for all } \omega_h \in \sigma(\mathcal{P}_h \mathcal{I}^* \mathcal{A}) \cap \Lambda,$$

where $\varkappa = \varkappa(\mathcal{A}, \omega)$ is the maximal length of a Jordan chain corresponding to ω. Further, for any $(\mathbf{u}_h, p_h) \in \ker \mathcal{P}_h \mathcal{I}^* A(\omega_h)$ with $\|(\mathbf{u}_h, p_h)\|_{\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)} = 1$ there exists a constant $C > 0$ such that

$$\inf_{(\mathbf{u},p) \in \ker \mathcal{A}(\omega)} \|(\mathbf{u}_h, p_h) - (\mathbf{u}, p)\|_{\mathcal{H}} \leq C \left(|\omega_h - \omega| + \max_{\substack{(\mathbf{v},q) \in \ker \mathcal{A}(\omega) \\ \|(\mathbf{v},q)\|_{\mathcal{H}} = 1}} \text{dist}((\mathbf{v}, q), \mathbf{S}_h(\Omega) \times S_h(\Gamma)) \right).$$

Proof. The assertions follows from [11, Thm.1] and [25, Thm. 4]. $\qquad\square$

Theorem 3.2. *Let $\omega \in \sigma(\mathcal{A})$ and let $\Lambda \subset \mathbb{C} \setminus \{0\}$ be as in Theorem 3.1. Assume that there exists a $\delta \in (0, 1]$ such that $G(\mathcal{A}, \omega), G(\mathcal{A}^*, \omega) \subset \mathbf{H}^{1+\delta}(\Omega) \times H_{\text{pw}}^{1/2+\delta}(\Gamma)$. Then there exists a $h_0 > 0$ and a constant $c > 0$ such that for all $0 < h \leq h_0$ we have*

$$|\omega - \omega_h| \leq c h^{2\delta/\varkappa(\mathcal{A}, \omega)} \qquad \text{for all } \omega_h \in \sigma(\mathcal{P}_h \mathcal{I}^* \mathcal{A}) \cap \Lambda.$$

Furthermore, for any $(\mathbf{u}_h, p_h) \in \ker \mathcal{P}_h \mathcal{I}^* \mathcal{A}(\omega_h)$ with $\|(\mathbf{u}_h, p_h)\|_{\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)} = 1$ there exists a constant $C > 0$ such that

$$\inf_{(\mathbf{u},p) \in \ker \mathcal{A}(\omega)} \|(\mathbf{u}_h, p_h) - (\mathbf{u}, p)\|_{\mathbf{H}^1(\Omega_S) \times H^{1/2}(\Gamma)} \leq C h^\alpha, \qquad \alpha = \min\{1, \delta/\varkappa(\mathcal{A}, \omega)\}.$$

Proof. The error estimates are a direct consequence of Thm. 3.1 and the approximation properties of $\mathbf{S}_h(\Omega_S)$ and $S_h(\Gamma)$, see [3, Sect. 4.4] and [18, Thm. 2.3]. $\qquad\square$

4 Numerical examples

The Galerkin eigenvalue problem (21) results in a nonlinear matrix eigenvalue problem of the form

$$A_h(\omega_h) \left(\frac{u}{p} \right) = 0, \tag{23}$$

where

$$A_h(\omega_h) = \begin{pmatrix} -\omega_h^2 \rho_S M_h^{\text{FEM}} + A_h^{\text{FEM}} - \rho_F \omega_h^2 N_h^\top V_h^{\text{BEM}}(k_h) N_h & N_h^\top (\frac{1}{2} M_h^{\text{BEM}} + K_h^{\text{BEM}}(k_h)) \\ (\frac{1}{2} M_h^{\text{BEM}} + (K_h^{\text{BEM}}(k_h))^\top) N_h & \frac{1}{\rho_F \omega_h^2} D_h^{\text{BEM}}(k_h) \end{pmatrix},$$

where M_h^{FEM} and A_h^{FEM} are the finite element mass and stiffness matrices and $V_h^{\text{BEM}}(k_h)$, M_h^{BEM}, $K_h^{\text{BEM}}(k_h)$, and $D_h^{\text{BEM}}(k_h)$ are the boundary element matrices, see, e. g., [21]. The matrix N_h corresponds to the application of the normal component on the boundary, $\mathbf{u}_{|\Gamma} \cdot \mathbf{n}$.

For the numerical solution of (23) we use the contour integral method [2]. This method is suitable for the extraction of all eigenvalues which lie inside of a predefined contour in the complex plane. An alternative approach for the numerical solution of the nonlinear eigenvalue problem (23) is presented in [4] which is based on a polynomial interpolation of $A_h(\omega_h)$.

Example 4.1 As first numerical example we consider as solid domain Ω_S the spherical shell $\Omega_S := \{\mathbf{x} \in \mathbb{R}^3 \colon 4.95 < \|\mathbf{x}\| < 5\}$ and the fluid domain $\Omega_F := \{\mathbf{x} \in \mathbb{R}^3 \colon \|\mathbf{x}\| > 5\}$. For this example analytical approximations of the eigenvalues are derived in [9, Chapt. 10]. The material constants for the shell are $E = 207 \cdot 10^9$, $\nu = 0.3$ and $\rho_S = 7669$. For the surrounding fluid, we choose $c = 1483.24$ and $\rho_S = 1000$. The eigenvalues of practical interest are those which are lying close to the real axis, since the imaginary part of an eigenvalue corresponds to the damping of the related eigenfunction in time. As domain of interest for the eigenfrequencies $f = \omega/(2\pi)$ we have chosen the strip $\{f \colon 1 < \mathrm{Re}(f) < 90, \, |\mathrm{Im}(f)| < 5\}$. In this domain two analytical approximations are given in [9, Chapt. 10]. The results of the approximations of these eigenvalues for different meshes are presented in Table 1. The approximations of the eigenvalues on the two finest mesh-levels match well with the analytical approximations.

h/dof	0.5/8794	0.25/36792	0.15/109455	anal. approx.
$f_{h,1}$	(58.19,-1.44)	(55.82,-1.18)	(55.65,-1.16)	
$f_{h,2}$	(58.26,-1.45)	(55.84,-1.18)	(55.66,-1.16)	
$f_{h,3}$	(58.50,-1.48)	(55.84,-1.18)	(55.66,-1.16)	56.02
$f_{h,4}$	(58.62,-1.50)	(56.03,-1.20)	(55.78,-1.18)	
$f_{h,5}$	(58.96,-1.54)	(56.04,-1.21)	(55.78,-1.18)	
$f_{h,6}$	(83.61,-1.00)	(71.47,-0.32)	(70.45,-0.31)	
$f_{h,7}$	(83.73,-1.03)	(71.53,-0.32)	(70.53,-0.31)	
$f_{h,8}$	(84.51,-1.08)	(71.63,-0.32)	(70.53,-0.31)	
$f_{h,9}$	(85.10,-1.14)	(71.63,-0.32)	(70.54,-0.31)	70.52
$f_{h,10}$	(85.47,-1.16)	(71.72,-0.33)	(70.60,-0.31)	
$f_{h,11}$	(85.94,-1.18)	(71.74,-0.33)	(70.61,-0.31)	
$f_{h,12}$	(87.96,-1.37)	(71.80,-0.34)	(70.62,-0.32)	

Table 1: Eigenvalues of A_h of Example 4.1 in the strip $\{f \colon 1 < \mathrm{Re}(f) < 90, \, |\mathrm{Im}(f)| < 5\}$.

The eigenfunctions corresponding to the eigenvalues f_1 to f_5 and f_6 to f_{12} are axis-symmetric, respectively. In Fig. 2, plots of the deformation of the spherical shell Ω_S induced by the eigenfunctions corresponding to $f_{h,1}$ and $f_{h,6}$ are given.

Example 4.2 For the second numerical example we use a simplified submarine model with length $12m$, diameter $2m$ and wall thickness $0.1m$. As material for the submarine model we haven chosen titan and for the fluid water. In Table 2 the approximations of the eigenvalues in the strip $\{f \colon 1 < \mathrm{Re}(f) < 120, \, |\mathrm{Im}(f)| < 5\}$ are given for different mesh-sizes. Note that the eigenvalue $f_{3,h}$ it is not an eigenvalue of the original eigenvalue problem (1) since $(\mathbf{u}_{h,3}, p_{h,3})$ is an approximation of $(\mathbf{0}, p)$ where p is an eigenfunction corresponding to the eigenvalue $k = 2\pi f_3/c_F$ of the Neumann Laplacian eigenvalue problem
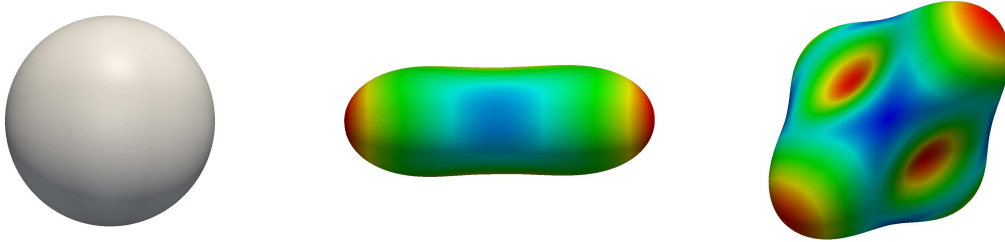
Figure 2: Undeformed spherical shell Ω_S and deformations of Ω_S induced by the eigenmodes corresponding to $f_{h,1}$ and $f_{h,6}$ for $h = 0.15$.
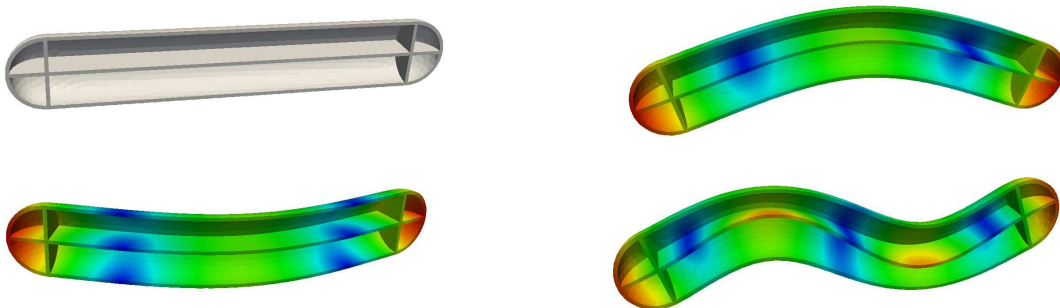

Figure 3: Undeformed submarine model Ω_S, where one half of the hull is not shown, and deformations induced by the eigenmodes corresponding to $f_{h,1}$, $f_{h,2}$ and $f_{h,4}$ for $h = 0.2$.

of the domain $\mathbb{R}^3 \setminus \overline{\Omega}_S$. Fig. 3 shows the deformations of the submarine model which are induced by the eigenmodes corresponding to the eigenvalues $f_{h,1}$, $f_{h,2}$ and $f_{h,4}$.

h/dof	0.4/4809	0.2/19114	0.1/74523
$f_{h,1}$	(54.18,-6.0e-3)	(52.41,-6.5e-3)	(52.12,-6.9e-3)
$f_{h,2}$	(59.11,-1.1e-2)	(57.48,-1.1e-2)	(57.25,-1.2e-2)
$f_{h,3}$	(65.71,-1.0e-8)	(65.53,-1.1e-8)	(65.48,-1.5e-8)
$f_{h,4}$	(126.6,-3.5e-1)	(119.9,-2.4e-1)	(118.6,-2.2e-1)

Table 2: Eigenvalues of A_h of Example 4.2 in the strip $\{f : 1 < \mathrm{Re}(f) < 120, \, |\mathrm{Im}(f)| < 5\}$.

5   Conclusions

In this paper, a coupled finite element and boundary element formulation for the eigenvalue problem of the solid-fluid interaction is proposed and analyzed. This formulation

is in particular suitable for the mid frequency regime since the acoustic pressure is modeled by the Helmholtz equation instead of the Laplace equation. The coupled eigenvalue problem is a nonlinear eigenvalue problem with respect to eigenvalue parameter, but it is holomorphic and fits in the framework of eigenvalue problems for holomorphic Fredholm operator-valued functions. Within this framework convergence and error estimates of the Galerkin discretization of the coupled eigenvalue problem have been derived. The contour integral method is a suitable method for the numerical solution of the discretized coupled eigenvalue problem. It provides all eigenvalues and corresponding eigenvectors which lie inside a predefined contour in the complex plane.

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