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Polygons and iteratively regularizing affine transformations

Otto Roeschel[1]

Abstract We start with a generic planar n-gon Q_0 with verities $q_{j,0}$ ($j = 0, \ldots, n - 1$) and fixed reals $u, v, w \in \mathbb{R}$ with $u + v + w = 1$. We iteratively define n-gons Q_k of generation $k \in \mathbb{N}$ with vertices $q_{j,k}$ ($j = 0, \ldots, n - 1$) via $q_{j,k} := u\, q_{j,k-1} + v\, q_{j+1,k-1} + w\, q_{j+2,k-1}$. We are able to show that this affine iteration process for general input data generally regularizes the polygons in the following sense: There is a series of affine mappings β_k such that the sums Δ_k of the squared distances between the vertices of $\beta_k(Q_k)$ and the respective vertices of a given regular prototype polygon P form a null series for $k \longrightarrow \infty$.

Keywords Affine Iterations · Affine Regularization · Regular n-gons

Mathematics Subject Classification 51N10 · 51N20

1 Introduction

Ziv (2002) and Donisi et al. (2016) studied a geometric iteration process starting with a generic planar polygon Q_0 with $n > 2$ vertices $\{q_{0,0}, \ldots, q_{n-1,0}\}$ ($q_{j,0} \in \mathbb{C}$). This process places homothetic copies of a given fixed triangle XYZ on the sides of Q_0. This way the third point Z of XYZ gets them the points of a *next generation polygon* Q_1. Reapplying this process over and over again with the same triangle XYZ creates a series of generations Q_k. Surprisingly, this iteration process, in general, has a regularizing effect on the polygon: The generation Q_k approaches the shape of a

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regular n-gon—see Donisi et al. (2016) and Ziv (2002). A general cyclic iteration process was introduced by Schoenberg (1950). He reports conditions and results on the existence of a limit point of these iterations. We will show that a regularizing effect generally shows up for affine iteration processes of this type. In order to prove this surprising property we set up an appropriate definition of 'affine regularization'.

2 The affine iteration process

We use complex numbers $z \in \mathbb{C}$ with $z = x + i\,y\,(x, y \in \mathbb{R})$ to describe points of the Euclidean plane $\mathbb{R}^2$ and start with a generic planar polygon Q_0 with $n > 2$ vertices $\{q_{0,0}, q_{1,0}, \ldots, q_{n-1,0}\}$ with $q_{j,0} \in \mathbb{C}$. Additionally, we choose a *reference triangle* abc and a reference point $z^* := u\,a + v\,b + w\,c$ with fixed barycentrics $(u, v, w) \in \mathbb{R}^3$ $(u + v + w = 1)$ with respect to abc.

We now define the affine mappings $\alpha_{j,1}$ from the ordered vertex set abc of the reference triangle to ordered triples of consecutive vertices $q_{j,0}, q_{j+1,0}, q_{j+2,0}$ of Q_0 ($j \in \mathbb{N}$; $j\ mod\ n$). The point z^* is mapped into points $\alpha_{j,1}(z^*) := q_{j,1} = u\,q_{j,0} + v\,q_{j+1,0} + w\,q_{j+2,0}$ ($j \in \mathbb{N}$, $j\ mod\ n$). These points form vertices of the *generation 1 polygon* Q_1. Iterations of this process yield affine mappings $\alpha_{j,k}$ and points

$$\alpha_{j,k}(z^*) : z^* = u\,a + v\,b + w\,c \longrightarrow \alpha_{j,k}(z^*) := q_{j,k}\ \text{with}$$
$$q_{j,k} = u\,q_{j,k-1} + v\,q_{j+1,k-1} + w\,q_{j+2,k-1}\ (k \in \mathbb{N} - \{0\}) \qquad (1)$$

($j \in \mathbb{N}$, $j\ mod\ n$) forming a series of polygons $Q_k := (q_{0,k}, q_{1,k}, \ldots, q_{n-1,k})$ called the kth *generation polygons*. This procedure generalizes the geometric iteration process presented and discussed in Radcliffe (2016). In the sequel Q_k also denotes the vectors $Q_k := (q_{0,k}, \ldots, q_{n-1,k})^t \in \mathbb{C}^n$. Figure 1 shows an example of the first iteration of procedure (1) for $n = 5$.

As we are interested in affine properties of the iteration process we describe affine mappings of the plane $\mathbb{R}^2$ by complex numbers: With $z = x + i\,y\,(x, y \in \mathbb{R}$, let the conjugate number to z be named $\bar{z} := x - i\,y)$ an affine mapping $\beta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $(x, y) \longrightarrow \beta(x, y) = (x^*, y^*) := (d_1 + a_1 x - b_2 y, d_2 + a_2 x + b_1 y)$ with $d_1, d_2, a_1, a_2, b_1, b_2 \in \mathbb{R}$. With the complex numbers $d := d_1 + i\,d_2, e := (a_1 + i\,a_2)/2$ and $f := (b_1 + i\,b_2)/2$ in $\mathbb{C}$ this yields

$$\beta : z \longrightarrow \beta(z) = d + z\,(e + f) + \bar{z}\,(e - f). \qquad (2)$$

We write $\beta(Q_k) := (\beta(q_{0,k}), \ldots, \beta(q_{n-1,k}))^t$. The affine mapping β is regular for $a_1 b_1 + a_2 b_2 \neq 0$ or, equivalently, $e\,\bar{f} + \bar{e}\,f \neq 0$.
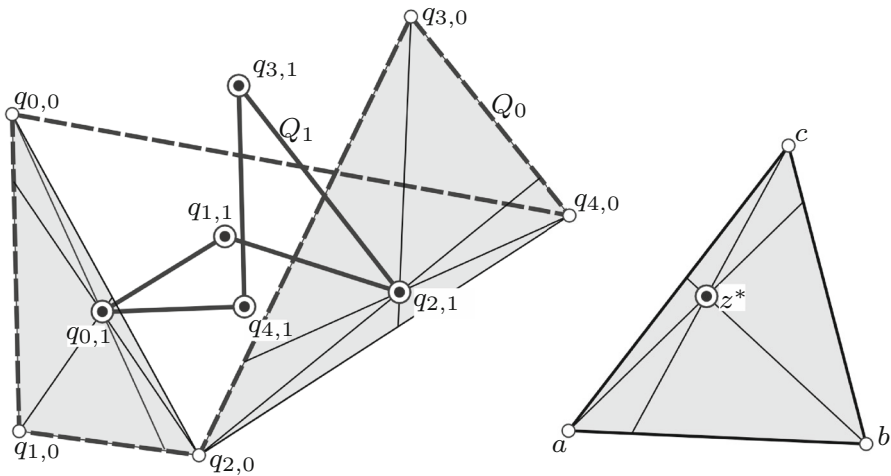
Fig. 1 The case $n = 5$: the polygon Q_0 and the first generation polygon Q_1 for $z^* = u\,a + v\,b + w\,c$

3 The iteration process and regular n-gons

Formula (1) can formally be written as

$$
\begin{pmatrix} q_{0,k} \\ \vdots \\ q_{n-1,k} \end{pmatrix} = \begin{pmatrix} u & v & w & 0 & \dots & 0 \\ 0 & u & v & w & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \dots & u & v & w \\ w & 0 & \dots & \dots & u & v \\ v & w & 0 & \cdots & 0 & u \end{pmatrix} \begin{pmatrix} q_{0,k-1} \\ \vdots \\ q_{n-1,k-1} \end{pmatrix} \Leftrightarrow
$$

$$
Q_k = M \cdot Q_{k-1}
$$

(3)

with the $n \times n$- matrix $M \in \mathbb{R}^{n,n}$. As M is a so-called *circulant* matrix, discrete Fourier transformations and the nth roots of unity $\zeta_j := exp(i\frac{2j\pi}{n}) = \cos\frac{2j\pi}{n} + i\sin\frac{2j\pi}{n}$ ($j \in \mathbb{Z}$) play a pivotal role for us (see Pech 2002, Schoenberg 1950 and Ziv 2002). For them we have $\zeta_j = \zeta_{j\,mod\,n}$, $\bar{\zeta}_j = \zeta_{n-j}$ ($j \in \mathbb{Z}$). We define the vectors

$$
P_j := (\zeta_j^0, \dots, \zeta_j^{n-1})^t \quad j \in \mathbb{Z}.
$$

(4)

We have $P_{n-j}^t M = (u\,\zeta_j^0 + v\,\zeta_j^1 + w\,\zeta_j^2)P_{n-j}^t$ and $M\,P_j = (u\,\zeta_j^0 + v\,\zeta_j^1 + w\,\zeta_j^2)P_j$ for all $j \in \mathbb{J} := \{0, \dots, n-1\}$. Therefore the vectors P_{n-j}^t and P_j are left and right eigenvectors of the matrix M to the same eigenvalue

$$
\lambda_j := u\,\zeta_j^0 + v\,\zeta_j^1 + w\,\zeta_j^2 \ (j \in \mathbb{J}).
$$

(5)

Owing to $(u, v, w) \in \mathbb{R}^3$ we have $\bar{\lambda}_j = \lambda_{n-j}$ for all $j \in \mathbb{J}$.
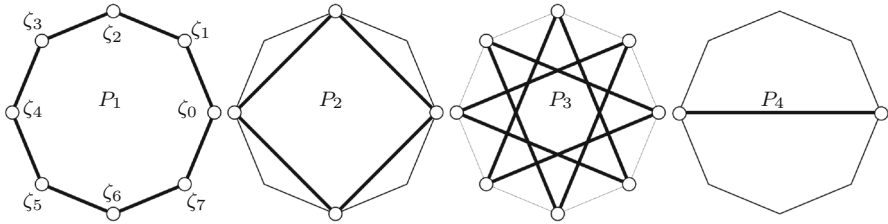
Fig. 2 The case $n = 8$: the regular octogons $P_1, \ldots P_4$ of first, second, third and forth kind

Remark 3.1 For $j \in \mathbb{J}^* := \{1, \ldots, n-1\}$ the elements of P_j can be interpreted as a collection of n points ζ_j^k ($k \in \mathbb{J}$) equally distributed on the unit circle. They determine vertices of a so-called *'regular n-gon of jth kind'*. The regular n-gon of $(n-j)$th kind P_{n-j} is the reflection of P_j in the real axis $y = 0$ and is affinely equivalent to P_j. If j and n are relatively prime the polygon P_j is either a regular n-gon or an n-sided regular star. If j is a divisor of n with $n = jm$ the polygon P_j either is a regular m-gon or a regular star with m vertices on the unit circle which has to be counted j times. For $j \in \mathbb{J}^*$ these regular n-gons P_j of jth kind share the origin 0 as their center of gravity. Figure 2 displays the situation for $n = 8$: The regular octogons P_7, P_6, P_5 coincide with P_1, P_2, P_3, but have opposite orientation.

4 The normal form of the matrix representation

We define two matrices

$$L := \frac{1}{\sqrt{n}} \, (P_n, \, P_{n-1}, \ldots, P_1) \quad \text{and} \quad R := \frac{1}{\sqrt{n}} \, (P_0, \, P_1, \ldots, P_{n-1}). \quad (6)$$

Both matrices, L and R, are symmetric and regular for $n > 1$ (see Pech 2002, Schoenberg 1950 and Ziv 2002). Moreover, we have $L = \bar{R}$ and $L \cdot R = I_{n,n}$ with the $n \times n$-unit matrix $I_{n,n}$; the matrices L and R are unitary $n \times n$-matrices in $\mathbb{C}^{n \times n}$. This way we have

$$L \cdot M \cdot R = D(\lambda_0, \ldots, \lambda_{n-1}) \quad (7)$$

with the diagonal matrix $D(\lambda_0, \ldots, \lambda_{n-1})$ containing the eigenvalues λ_j of M as elements of the main diagonal. We now have

$$
\begin{aligned}
L \cdot Q_k &= D(\lambda_0, \ldots, \lambda_{n-1}) \cdot L \cdot Q_{k-1} \text{ and} \\
L \cdot Q_k &= D(\lambda_0, \ldots, \lambda_{n-1})^k \cdot L \cdot Q_0 \quad (k \in \mathbb{N} - \{0\}).
\end{aligned}
\quad (8)
$$

For the jth component of the vectors in (8) this can be written as

$$\sum_{m=0}^{n-1} \zeta_{n-j}^{n-m} q_{m,k} = \lambda_j^k \sum_{m=0}^{n-1} \zeta_{n-j}^{n-m} q_{m,0} = (u\zeta_j^0 + v\zeta_j^1 + w\zeta_j^2)^k \sum_{m=0}^{n-1} \zeta_{n-j}^{n-m} q_{m,0} \quad (9)$$

for all $k \in \mathbb{N} - \{0\}$ and $j \in \mathbb{J}$. The element $j = 0$ yields $\lambda_0^k = (u+v+w)^k = 1 \, \forall k \in \mathbb{N}$ and therefore $\sum_{m=0}^{n-1} q_{m,k} = \sum_{m=0}^{n-1} q_{m,0}$ for all $k \in \mathbb{N} - \{0\}$. Thus, all polygons Q_k have the same center of gravity ($k \in \mathbb{N}$). Without loss of generality, we put this center of gravity into the origin 0 of our coordinate system. Thus, we assume that the initial polygon Q_0 has its center of gravity in 0:

$$\frac{1}{n} \sum_{m=0}^{n-1} q_{m,0} = 0. \tag{10}$$

As the matrix L is regular the initial polygon Q_0 can be explicitly recovered from the vector

$$B := L \, Q_0 \;\; \text{with} \; B = (0, b_1, \dots, b_{n-1})^t \in \mathbb{C}^{n-1}. \tag{11}$$

Under our assumption (10) the first coordinate of the vectors in (8) is 0. Thus, we do not alter the recursion if we replace the matrix $D(\lambda_0, \dots, \lambda_{n-1})$ in (8) by the diagonal matrix $D^* := D(0, \lambda_1, \dots, \lambda_{n-1})$. With that and (11) in mind, the iteration process is described by

$$L \cdot Q_k = D^{*k} \cdot B \;\; \Leftrightarrow \;\; Q_k = \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{J}^*} \lambda_j^k \, b_j \, P_j \;\; (k \in \mathbb{N}). \tag{12}$$

5 Definition of affine regularization

Just like the iterations defined in Donisi et al. (2016) and Ziv (2002), iteration (1) also seems to regularize 'in a way' for certain $(u, v, w) \in \mathbb{R}^3$ independent of the choice of the initial polgon Q_0. In order to discuss this interesting property we compare the n-gons Q_k with a regular prototype n-gon P_j of kind j.

Definition 5.1 We call the procedure (1) *affinely regularizing of kind* $j \in \mathbb{J}^*$ if, for general input data (initial polygon Q_0), there exist affine mappings β_k transforming $Q_k = (q_{0,k}, \dots, q_{n-1,k})^t$ into polygons $\beta_k(Q_k)$ with the property that for all $k \in \mathbb{N}$ the series Δ_k of sums of the squared distances of the points of $\beta_k(Q_k)$ to the regular prototype polygon P_j (4) of kind j is a null series.

Thus, the iterative processes (1) is affinely regularizing of kind j if there exist affine mappings β_k with:

$$\Delta_k := \sum_{m=0}^{n-1} dist^2(\beta_k(q_{m,k}), \zeta_j^m) = (P_j - \beta_k(Q_k))^t \cdot \overline{(P_j - \beta_k(Q_k))} \tag{13}$$

has the property $\lim_{k \to \infty} \Delta_k = 0$.

Remark 5.2 – It is important to note that definition 5.1 does not require the existence of a limit polygon Q_k for $k \to \infty$. In general for the iterative processes (1) such a limit polygon will not exist. 'Affinely regularizing' in our sense is a property of the affine shape of the series Q_k alone.

– In order to be able to measure the sum of the squared distances in Definiton 5.1 we need the Euclidean structure of the plane.

– Our definiton makes use of affine mappings β_k transforming Q_k 'close to some regular prototype polygon' P_j of fixed size. In order to define the regulization of the iteration process it would not be enough to do it the other way round: the existence of affine mappings γ_k transforming the prototype polygon P_j close to the polygons Q_k (again with sums of squared distances forming a null series) falls short of our requirements. The following counterexample demonstrates that fact: Given a series of polygons $Q_k := (1/2)^k Q_0$ that are homothetic copies of an initial polygon Q_0. For any $j \in \mathbb{J}^*$ the affine mappings $\gamma_k(P_j) := (1/2)^k P_j$ yield sums of squared distances of corresponding vertices of Q_k and $\gamma_k(P_j)$ forming a null series independent from the input data Q_0. As Q_k is homothetic to a polygon Q_0 'of general shape' this process, in general, is far from being affinely regularizing.

6 The affine shape of Q_k

The shape of the polygons Q_k depends on the input data set Q_0 and the barycentric corordinates (u, v, w) of the point z with $u + v + w = 1$. (u, v, w) uniquely determine the diagonal matrix $D^* = D(0, \lambda_1, \ldots, \lambda_{n-1})$. We compute the norms $n_j := \sqrt{\lambda_j \bar{\lambda}_j}$ of the eigenvalues λ_j for $j \in \mathbb{J}^*$ and have

$$n_j^2 = \lambda_j \bar{\lambda}_j = u^2 + v^2 + w^2 + v(u + w)(\zeta_j + \zeta_{-j}) + uw(\zeta_j^2 + \zeta_{-j}^2)$$
$$= u^2 + v^2 + w^2 + 2\,v(u + w)\cos(2\pi j/n) + 2\,uw\cos(4\pi j/n) \qquad (14)$$

$(j \in \mathbb{J}^*)$. The *spectral radius of the matrix D^** is the maximal value $N := \max\{n_1, \ldots, n_{n-1}\}$ of the norms of the eigenvalues λ_j for $j \in \mathbb{J}^*$. We assume that the barycentrics (u, v, w) with $u + v + w = 1$ are chosen *generally* in the sense that not all $\lambda_1, \ldots, \lambda_{n-1}$ vanish.

It is easy to see that the *exceptional case* $\lambda_1 = \cdots = \lambda_{n-1} = 0$ can only occour in the case where $n = 3$ and, additionally, $(u, v, w) = (1/3, 1/3, 1/3)$. Therefore, this case in general yields iterated series of 'degenerate triangles' Q_k that are formed by the center of gravity 0. This *trivial case* shall be excluded from now on by putting $n > 3$.

In all other cases we have that $N > 0$. So we can define the two sets of indices

$$\mathbb{J}_1 := \{j \in \mathbb{J}^*/ |\lambda_j| = N\} \neq \emptyset \quad \text{and} \quad \mathbb{J}_2 := \mathbb{J}^* - \mathbb{J}_1. \qquad (15)$$

Note that, according to (5), for any $j^* \in \mathbb{J}_1$ the index $n - j^*$ is also contained in $\mathbb{J}_1$ (only for even n and $j^* = n/2$ these two indices are coincident) and we have

$$\frac{|\lambda_j|}{N} = 1 \; \forall j \in \mathbb{J}_1, \quad \frac{|\lambda_j|}{N} < 1 \; \forall j \in \mathbb{J}_2. \qquad (16)$$

Figure 3 shows the positions of λ_j for $n = 8$ and the barycentrics (u, v, w) of z^* with respect to the triangle abc. For this special case we have $\mathbb{J}_1 = \{1, 7\}$.
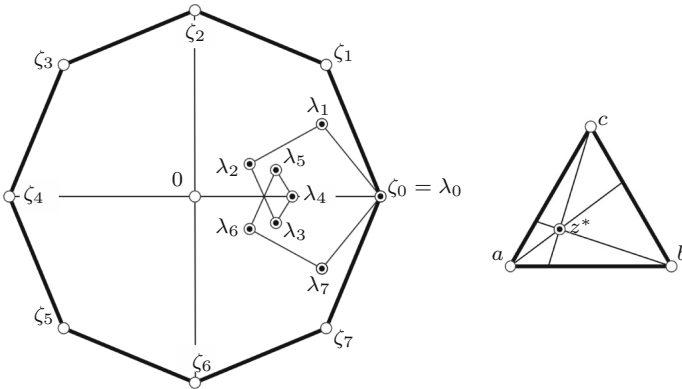
Fig. 3 The case $n = 8$: the eigenvalues λ_j for $n = 8$ and $z^* = u\,a + v\,b + w\,c$

Equation (12) yields

$$Q_k = \frac{N^k}{\sqrt{n}} \left(\sum_{j \in \mathbb{J}_1} (\frac{\lambda_j}{N})^k \, b_j \, P_j + \sum_{j \in \mathbb{J}_2} (\frac{\lambda_j}{N})^k \, b_j \, P_j \right). \tag{17}$$

Independent of the input data b_j the coefficients $(\frac{\lambda_j}{N})^k b_j$ of P_j in the second sum in (17) form null series for all $j \in \mathbb{J}_2$ and $k \longrightarrow \infty$. The coefficients in the first sum are finite complex numbers for all $k \in \mathbb{N}$. Depending on the spectral radius N formula (17) delivers three possibilities:

– For $N < 1$ the series Q_k decreases for increasing k and tends towards the center of gravity 0.
– For $N = 1$ the series Q_k remains finite, but in general changes its position for different k.
– For $N > 1$ the series Q_k increases for increasing k and seems to explode.

For all these possibilities, though, the series Q_k seems to be affinely regularizing. This is to be discussed in three cases:

Case A The index set $\mathbb{J}_1$ contains exactly one element.

As stated before, this is only possible for even n and special barycentrics (u, v, w) such that $\mathbb{J}_1 = \{n/2\}$, $\mathbb{J}_2 = \mathbb{J}^* - \{n/2\}$. We have $\zeta_{n/2} = -1$, and therefore $\lambda_{n/2} = u - v + w \in \mathbb{R}$. This yields $|\lambda_{n/2}|/N = (-1)^s$ with $s \in \{0, 1\}$ depending on the sign of $u - v + w$. The only element in the first sum of (17) has the coefficient $c_k^* := (-1)^{sk} b_{n/2}$. For a general input polygon Q_0 we have $b_{n/2} \neq 0$ and therefore $c_k^* \neq 0$. We can define the following affine image of the kth generation Q_k:

$$\beta_k(Q_k) := \frac{\sqrt{n}}{c_k^* N^k} Q_k = P_{n/2} + \frac{1}{c_k^*} \sum_{j \in \mathbb{J}_2} \left(\frac{\lambda_j}{N}\right)^k b_j P_j. \tag{18}$$

With (17) we compute the sum of squared distances of $\beta_k(Q_k)$ from $P_{n/2}$ via

$$\Delta_k = n \left(\sum_{j \in \mathbb{J}_2} (\tfrac{\bar{\lambda}_j \lambda_j}{N^2})^k \, \bar{b}_j \, b_j \right) / (\bar{b}_{n/2} \, b_{n/2}). \tag{19}$$

As stated before, for all $j \in \mathbb{J}_2$ the value $(\tfrac{\bar{\lambda}_j \lambda_j}{N^2})^k$ decreases for increasing k and $\lim_{k \longrightarrow \infty} (\tfrac{\bar{\lambda}_j \lambda_j}{N^2})^k = 0$. Thus, according to our Definition 5.1 the iteration process in Case A for general input data is affinely regularizing of kind $n/2$. For large k generation Q_k will approach the shape of the degenerate regular n-gon $P_{n/2}$: It consists of two points, each of them counted $n/2$ times.

Case B The index set $\mathbb{J}_1$ contains exactly two different elements: $\mathbb{J}_1 = \{j^*, n - j^*\}$ with $j^* \neq n/2$ (in Fig. 3 we have an example for $n = 8$ with $j^* = 1$). We put $\lambda_{j^*} = N e^{i \phi}$ and $\lambda_{n-j^*} = N e^{-i \phi}$ with some real angle $\phi \in [0, 2\pi)$. Then (17) yields

$$Q_k = \frac{N^k}{\sqrt{n}} \left(e^{i k \phi} \, b_{j^*} \, P_{j^*} + e^{-i k \phi} \, b_{n-j^*} \, \bar{P}_{j^*} + \sum_{j \in \mathbb{J}_2} \left(\frac{\lambda_j}{N} \right)^k b_j \, P_j \right). \tag{20}$$

We define $\sigma := b_{j^*} \bar{b}_{j^*} - b_{n-j^*} \bar{b}_{n-j^*}$ ($\sigma \in \mathbb{R}$). For general input data we have $\sigma \neq 0$ and we can define affine images of the kth generation Q_k:

$$\begin{aligned}
\beta_k(Q_k) &:= \frac{\sqrt{n}}{\sigma \, N^k} e^{-ik\phi} \left(\bar{b}_{j^*} \, Q_k - b_{n-j^*} \, \bar{Q}_k \right) \\
&= P_{j^*} + \frac{1}{\sigma} e^{-ik\phi} \sum_{j \in \mathbb{J}_2} (\tfrac{\lambda_j}{N})^k \, (b_j \bar{b}_{j^*} - \bar{b}_{n-j} b_{n-j^*}) \, P_j
\end{aligned} \tag{21}$$

Now we compute the sum $\Delta_k := (P_{j^*} - \beta_k(Q_k))^t \cdot \overline{(P_{j^*} - \beta_k(Q_k))}$ of squared distances of respective points of $\beta_k(Q_k)$ and P_{j^*}. Short computation yields

$$\Delta_k = \frac{n}{\sigma^2} \sum_{j \in \mathbb{J}_2} \left(\frac{\bar{\lambda}_j \lambda_j}{N^2} \right)^k (b_j \bar{b}_{j^*} - \bar{b}_{n-j} b_{n-j^*}) \, (\bar{b}_j b_{j^*} - b_{n-j} \bar{b}_{n-j^*}). \tag{22}$$

The only factors in (22) depending on k are $(\tfrac{\bar{\lambda}_j \lambda_j}{N^2})^k$. For each $j \in \mathbb{J}_2$ these terms form null series for $k \in \mathbb{N}$. The finite sum Δ_k of null series delivers a null series again and we have $\lim_{k \longrightarrow \infty} \Delta_k = 0$. Thus, the corresponding iteration process is affinely regularizing of kind j^*.

Case C The index set $\mathbb{J}_1$ contains more than two different elements $j^*, j^{**}, n - j^*$ with $j^{**} \neq j^*$ and $j^{**} \neq n - j^*$. According to (14) we have

$$\begin{aligned}
v(u + w)(\zeta_{j^*} + \zeta_{-j^*} - \zeta_{j^{**}} - \zeta_{-j^{**}}) \\
+ uw(\zeta_{j^*}^2 + \zeta_{-j^*}^2 - \zeta_{j^{**}}^2 - \zeta_{-j^{**}}^2) = 0.
\end{aligned} \tag{23}$$

In general this is the equation of a conic section in the plane of the triangle abc. For these special choices of the barycentrics (u, v, w) we cannnot give any affine mappings like in Cases A and B for general input polygon Q_0 which deliver a null series for the corresponding distance functions.

We have:

Theorem 6.1 *For a general input polygon Q_0 and general barycentrics (u, v, w) the iteration process (1) is affinely regularizing.*

Remark 6.2 In general, the shape of Q_k gradually approaches that of an affinely transformed regular n-gon P_{j*} of kind j^*. The kind j^* of this affine regularization belongs to those indices j^* which determine the spectral radius $N = |\lambda_{j*}|$ of the transformation matrix $D^* = D(0, \lambda_1, \ldots, \lambda_{n-1})$.

The exceptional cases are addressed in

Theorem 6.3 *In the following exceptional cases the iteration process (1), in general, fails to regularize:*

- *Case A We have even n, special barycentrics (u, v, w) and get case A with $c_k^* = 0$. The initial polygon Q_0 has a special shape with $b_{n/2} = 0$.*
- *Case B The input polygon Q_0 has a special shape with $\sigma := b_{j*}\bar{b}_{j*} - b_{n-j*}\bar{b}_{n-j*} = 0$.*
- *Case C The input barycentrics $(u, v, w) \in \mathbb{R}^3$ with $u + v + w = 1$ comply with (23) for suitable triples $(j^*, j^{**}, n - j^*)$ consisting of three different indices from the index set $\mathbb{J}_1$.*

Remark 6.4 As long as we are not in the realm of the exceptional cases listed in Theorem 6.3 the regularizing property of Theorem 6.1 remains valid, even if some of the data points of the initial polygon Q_0 or of some further generation are coincident.

Figure 4 shows an example for $n = 7$ with the barycentrics $(u, v, w) = (0.5, -0.2, 0.7)$ of z^*. We get $(n_1^2, n_2^2, n_3^2) \approx (0.57, 0.51, 1.28)$ and therefore we have $\mathbb{J}_1 = \{3, 4\}$. This way we have an example for case B. The figure shows Q_0 and the 10th generation Q_{10} which approaches the affine image of the regular star 7-gon P_3.

7 Generalisations

The considerations from above can be modified to obtain further affinely regularizing iteration processes. Instead of the planar affine mappings used in (1) we can also use affine mappings from an n-dimensional affine space with basic points in the vertices of a simplex $a_0, \ldots, a_{n-1}$. A point $z^* := \sum_{\mu=0}^{n-1} u_\mu a_\mu$ with barycentrics $(u_0, \ldots, u_{n-1})$ and $\sum_{\mu=0}^{n-1} u_\mu = 1$ with respect to that simplex then is mapped to the plane of the initial polygon Q_0 via affine mappings
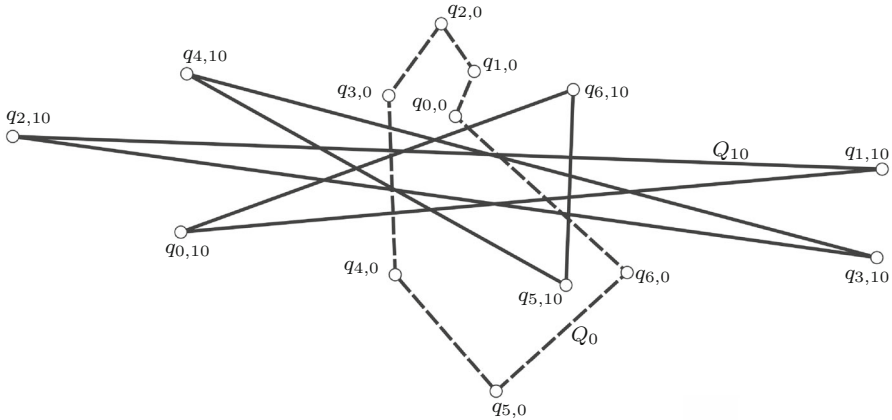
Fig. 4 The case $n = 7$ with $\mathbb{J}_1 = \{3, 4\}$

$$\alpha_{j,k}(z^*) : z^* = \sum_{\mu=0}^{n-1} u_\mu a_\mu \longrightarrow q_{j,k} \text{ with}$$

$$q_{j,k} := \sum_{\mu=0}^{n-1} u_\mu q_{j+\mu(mod\,n),k-1} \ (j \in \mathbb{J}; \ k \in \mathbb{N} - \{0\}). \qquad (24)$$

Such an iteration process has been studied by Schoenberg (1950). The author did not state results on the affine shape of Q_k. In this case the eigenvalues λ_j are given by

$$\lambda_j := \sum_{\mu=0}^{n-1} u_\mu \zeta_j^\mu \ (j \in \mathbb{J}). \qquad (25)$$

These generalized iteration processes show the same behavior as that described in Theorems 6.1 and 6.3. Again, for general input data Q_0 and general u_μ ($\mu = 0, \ldots, n-1$) with $\sum_{\mu=0}^{n-1} u_\mu = 1$ the corresponding iteration processes are still affinely regularizing. Our results could be viewed as a nice add-on to I. J. Schoenberg's paper (Schoenberg 1950).

References

Donisi, S., Martini, H., Vincenzi, G., Vitale, G.: Polygons derived from polygons via iterated constructions. Electron. J. Differ. Geom. Dyn. Syst. **18**, 14–31 (2016)

Pech, P.: The harmonic analysis of polygons and Napoleon's theorem. J. Geom. Gr. **5**(1), 13–22 (2001)

Radcliffe, D.: Inscribed polygons and the Fourier transform. https://mathblag.wordpress.com/2013/10/08/inscribed-polygons-and-the-fourier-transform/ (2016)

Schoenberg, I.J.: The finite Fourier series and elemenatry geometry. Am. Math. Monthly **57**, 390–404 (1950)

Ziv, B.: Napoleon-like configurations and sequences of triangles. Forum Geom. **2**, 115–128 (2002)