Real-time dynamics in a strongly interacting bosonic hopping model: Global quenches and mapping to the XX chain

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March 30, 2016

Abstract

We study the time evolution of an integrable many-particle system, described by the q-boson Hamiltonian in the limit of strong bosonic interactions $q \to \infty$. It is shown that, for a particular class of pure initial states, the analytical calculation of certain observables simplifies considerably. Namely, we provide exact formulas for the calculation of the Loschmidt-echo and the emptiness formation probability, where the computational time scales polynomially with the particle number. Moreover, we construct a non-local mapping of the q-boson model to the XX spin chain, and show how this can be utilized to obtain the time evolution of various local bosonic observables for translationally invariant initial states. The results obtained via the bosonic and fermionic picture show perfect agreement. In the infinite volume and large time limits, we rigorously verify the prediction of the Generalized Gibbs Ensemble for homogeneous initial Fock states.

1 Introduction

The study of time evolution in isolated quantum many-body systems has become a leading direction of research within both experimental and theoretical condensed matter physics [1, 2]. One of the main questions to be answered is whether unitary time evolution can lead to a local thermalization of the system and what are the characteristics of the relaxation towards this stationary state. Particularly interesting is the case of integrable quantum systems where the dynamics is severely constrained by an extensive set of conservation laws. In order to take these constraints into account, an extended notion of thermalization is required and a statistical ensemble under the name of Generalized Gibbs Ensemble (GGE) was put forward for the description of the stationary state [3].

One way of testing the validity of the GGE is by monitoring the asymptotic time evolution of local observables, which is a notoriously hard task for genuinely interacting Hamiltonians. One possibility is to resort to purely numerical techniques such as the time evolving block decimation method [4, 5] applied in [6], or to develop special tools based on integrability. From the analytical side, one of the most successful approaches to give predictions for the long time limit of observables is the so-called quench action technique [7]. It provides a means of capturing the stationary state emerging from the unitary time evolution of a Bethe Ansatz integrable model from a pure initial state. The method has been successfully applied for some simple initial states of the 1D Bose gas [8] and the infinite XXZ chain [9, 6]. Remarkably, in the latter case it has been found that the stationary state can only be described by a GGE that is supplemented by a new family of quasi-local conserved charges [10]. Recently, some generalizations to finite size chains have also been reported [11].

Despite the success of the Quench Action method, its scope is mainly restricted to the asymptotic regime of the time evolution. Indeed, the technique relies on selecting a representative Bethe state of the interacting Hamiltonian via a saddle point analysis, which in turn encodes all the information about the time-evolved state for $t \to \infty$. However, if one is interested in early or intermediate time-scales, the summation over the complete set of Bethe states, contributing to the time evolution of a specific observable, can not be avoided. In order to overcome the limitations of summing over an exponentially large set of eigenstates, one needs some prior knowledge about the importance of the contributions from many-particle states, which is a crucial ingredient behind Bethe Ansatz based numerical methods [12, 13]. Although such an importance sampling is feasible for some integrable models [14], it is not yet clear whether strict analytical results could be obtained using this technique.

On the other hand, within the field of Bethe Ansatz integrable systems, there has always been an immense theoretical interest in devising exact analytical techniques. Indeed, several successful approaches exist for the computation of equilibrium properties of these models, in particular the Heisenberg XXZ chain and the 1D Bose gas [15]. Important achievements include the computation of the asymptotics of space and time dependent correlation functions (see [16] and references therein) and the derivation of effective, factorized formulas for the ground state and finite temperature local correlators of the XXZ model (see [17] and references therein). In view of these remarkable achievements the question naturally arises whether some progress can be made for the far-from-equilibrium physics of integrable systems. Such studies are also motivated by the fact that in certain quenches of the XXZ chain and the 1D Bose gas exact results have been obtained for the stationary states [8, 9, 18, 19, 10, 20]. Ideally, one would like to have exact formulas for the full time evolution, such that the previously mentioned results could be obtained simply by taking the long time limit. Whether or not such program can be carried out is not clear at the moment.

Motivated by these long term goals, in the present paper we set a somewhat simpler objective: we consider time evolution in the q-boson Hamiltonian in the $q \to \infty$ limit, also known as the phase model [21, 22, 23, 24]. In this limit the bosonic interactions are strong, there is non-trivial scattering between the particles, but the scattering is simple enough so that manageable exact expressions can be obtained for the observables. Therefore, the complexity of this lattice hopping model is somewhere between that of free theories and a generic Bethe Ansatz solvable model. The quantities of interest are the return probability (or Loschmidt echo) and the emptiness formation probability (EFP) of the time evolved state. The EFP is probably the simplest local observable for which efficient closed-form expressions can be found for integrable models [25, 26]. On the other hand, the calculation of the Loschmidt echo mainly serves as an introductory example to demonstrate our method. Namely, we show that, due to the simple constraints between rapidities and for a specific class of initial states, the sums over exponentially many Bethe states can be turned into a simple sum over the total momentum and an auxiliary variable. Hence, we obtain an exact analytical expression for the Loschmidt echo where the number of terms to be summed scales at most linearly in both the system size and the number of particles. Interestingly, it turns out that the form factors of the EFP have again the properties which allow the same trick to be carried out, i.e. the exponential sums over intermediate states can again be replaced by an expression in which the number of terms scales polynomially.

The huge simplification in the computational efforts of the above quantities suggests that there might be some deeper connection between the q-boson Hamiltonian and a non-interacting model. In fact, such a mapping to the XX spin chain (which is equivalent to a free-fermion hopping model) was already pointed out in an earlier work [24]. Here we show that, although some boundary terms spoil an exact correspondence, the mapping can be symmetrized such that the two models become equivalent in the zero-momentum sector even for finite chain sizes. Furthermore, even though the mapping is non-local, we show that the average EFP's are exactly mapped onto each other. Thus, for translationally invariant bosonic initial states, the calculation of the time evolution of the EFP further simplifies using the free-fermion representation. Moreover, the fermionic methods allow us to compute other observables (such as the local bosonic occupation probabilities) that were previously inaccessible with the bosonic approach.

In the following, we first introduce the q-boson model in section 2 and describe its Bethe Ansatz solution in the limit $q = \infty$. In section 3 we report our main results about the

analytical calculation of the time evolution of the Loschmidt echo and the EFP for a simple class of initial states. Sec. 4 describes the non-local mapping from the q-boson Hamiltonian to the XX chain and introduces the free-fermion formalism for the calculation of the EFP. Some particular translation invariant states are considered in Sec. 5 where the numerical results on the EFP from our new Bethe Ansatz approach are cross-checked to analytical formulas obtained via the fermionic representation. Our concluding remarks are found in Sec. 6. The paper is supplemented with two appendices where some details of the analytical calculations are given.

2 The model and its Bethe Ansatz solution

Consider a bosonic chain of length L. The Hilbert space is spanned by Fock states

$$|n_1, n_2, \dots, n_L\rangle = |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots \otimes |n_L\rangle_L, \tag{2.1}$$

where $n_j \geq 0$ represent the local occupation numbers.

The q-boson model at $q = \infty$ (also called the phase model) is given by the Hamiltonian [1]

$$H_{\mathrm{B}} = -\sum_{j=1}^{L} (\phi_j^\dagger \phi_{j+1} + \phi_{j+1}^\dagger \phi_j - 2N_j), \tag{2.2}$$

where the operators ϕ_j, $\phi_j^\dagger$ are defined by their action

$$\phi_j |n\rangle_j = \begin{cases} |n-1\rangle_j, & \text{for } n > 0 \\ 0, & \text{for } n = 0, \end{cases} \qquad \text{and} \qquad \phi_j^\dagger |n\rangle_j = |n+1\rangle_j \tag{2.3}$$

on local bosonic states and the N_j are the standard local number operators. In (2.2) periodic boundary conditions are assumed. Note that the ϕ, $\phi^\dagger$ operators do not coincide with the standard bosonic creation/annihilation operators, in particular they satisfy the somewhat unusual exchange relation

$$[\phi_j, \phi_k^\dagger] = \delta_{j,k} \delta_{n_j,0}.$$

Although there is no explicit interaction term in the Hamiltonian (2.2), it is not free: due to definition (2.3) the physical hopping amplitudes depend on the local occupation numbers.

The q-boson model can be solved by the different versions of the Bethe Ansatz. The Algebraic Bethe Ansatz (ABA) solution was first derived (for general q) in the papers [28, 29, 30], whereas equilibrium correlation functions were calculated in [21]. Afterwards, the coordinate Bethe Ansatz wave functions were given in [22, 24]. The phase model is also related to the enumeration of plane partitions [22, 23, 24, 31].

In the present work we refrain from discussing the ABA solution as all our results can be obtained from the known real space wave functions. We write the N-particle eigenstates of the system as

$$|\{a\}_N\rangle = \sum_{1 \leq x_1 \leq x_2 \leq \dots x_N \leq L} C_N(x_1, \dots, x_N) \phi_{x_1}^\dagger \dots \phi_{x_N}^\dagger |0\rangle_{\mathrm{B}}. \tag{2.4}$$

It was derived in [22] that the coefficients can be expressed as

$$C_N(x_1, \dots, x_N) = \det_N \left((a_k)^{j+x_j} \right), \qquad a_k = e^{ip_k}, \tag{2.5}$$

where the p_k variables in (2.5) can be identified as quasi-momenta of the interacting particles. We note that our formula (2.5) differs from the conventions of the ABA literature in both the overall norm and the phase of the vector. In particular, there is a sign depending on the ordering of the parameters a_j, however this drops out from the actual calculations. Also, we will assume for simplicity that N and L are even. Odd values only affect certain signs in intermediate results, but not the physical observables.

[1] In the present work we do not treat the model for general q. The reader who is interested in the general case is referred to [27] and references therein.

In a Bethe Ansatz wave function a simple phase arises when we exchange two particle positions, and this is interpreted as the physical S-matrix of the particles. In the present case we can read off (2.5) that this phase is

$$S(p_1, p_2) = -e^{i(p_1 - p_2)}. \tag{2.6}$$

Periodicity of the wave function implies the Bethe equations:

$$e^{ip_j L} \prod_{k \neq j} S(p_j - p_k) = 1.$$

In the phase model this gives

$$(a_j)^{L+N} = (-1)^{N-1} e^{iP}, \qquad e^{iP} = \prod_{k=1}^{N} a_k, \tag{2.7}$$

where e^{iP} is the eigenvalue of the one-shift translation operator. Equation (2.7) shows that the quantization condition is "almost free": once P is fixed the momenta can be chosen independently, as long as they satisfy the overall constraint on the r.h.s. of (2.7). However, the exponent of the variable a_j is $L + N$, whereas a free theory would give simply L.

If the BA equations are satisfied, the energy eigenvalues are given by

$$E_N = \sum_{j=1}^{N} e(p_j) \quad \text{where} \quad e(p) = 4 \sin^2(p/2). \tag{2.8}$$

The norm of the Bethe states is simply

$$\langle \{a\}_N | \{a\}_N \rangle = L(L+N)^{(N-1)}. \tag{2.9}$$

This result was obtained in [21] using the Algebraic Bethe Ansatz, whereas in [22] it was shown that it follows from certain properties of the Schur polynomials [2].

In the thermodynamic limit $N, L \to \infty$ and $N/L = n$ it is convenient to work with root and hole densities $\rho_r(p)$ and $\rho_h(p)$. In generic Bethe Ansatz solvable models these densities satisfy linear integral equations, however in the present case the constraint for them reads simply

$$\rho_r(p) + \rho_h(p) = 1 + n, \tag{2.10}$$

which can be obtained directly from the Bethe equations (2.7).

3 Finite volume real time dynamics

In this section we present a new numerical method to study real time dynamics in the phase model. Our method is similar to that of [21], where two-point functions were computed in equilibrium.

In all cases below the setting is the following. At $t = 0$ the system of finite volume L is prepared in a state $|\Psi_0\rangle$, which can be a simple Fock state in the coordinate basis, or some other state prepared according to certain rules. At times $t > 0$ the system is subject to time evolution governed by the Hamiltonian (2.2) and our goal is to compute the physical observables.

We consider a class of initial states where the unnormalized overlaps can be written as

$$\langle \Psi_0 | \{a\}_N \rangle = Z(N) \det G_j(a_k), \tag{3.1}$$

where $Z(N)$ is a numerical constant, and G is an $N \times N$ matrix where the kth column only depends on the parameter a_k. It is easy to see that pure Fock states in the coordinate basis satisfy this requirement: the overlap can be read off from (2.5) and we get

$$G_j(a_k) = a_k^{x_j + j}, \qquad Z(N) = 1. \tag{3.2}$$

In the calculations below we will keep G unspecified as long as possible, but in the actual examples we will deal with G-matrices of the form (3.2).

[2] In [21, 22] the norm formula includes a Vandermonde determinant, which arises due to the different normalization of the Bethe vectors.

3.1 Exact result for the Loschmidt-echo

As a warm up we calculate the Loschmidt amplitude (or fidelity), which is defined as

$$\mathcal{L}(t) \equiv \langle \Psi_0 | \Psi(t) \rangle = \langle \Psi_0 | e^{-iH_B t} | \Psi_0 \rangle. \tag{3.3}$$

Inserting a complete set of Bethe states and using the assumed form (3.1) we have

$$\mathcal{L}(t) = \frac{Z^2(N)}{L(L+N)^{N-1}} \sum_{\{a\}} |\det G_j(a_k)|^2 e^{-iE(a)t}.$$

Our goal is to perform the summation over the a-variables without explicitly using the functions $G_j(a)$. First we give an explicit solution of the Bethe equations (2.7). Overall momentum quantization gives

$$P = J\frac{2\pi}{L}, \qquad J = 1, \ldots, L, \tag{3.4}$$

and the a-variables can then be parametrized as

$$a_j = e^{ip_j}, \qquad p_j = I_j \frac{2\pi}{N+L} + \frac{\pi + P}{N+L} \qquad I_j \in \{1, \ldots, N+L\}, \tag{3.5}$$

where the I_j are interpreted as momentum quantum numbers. These parameters can not be chosen completely freely, because they have to satisfy a constraint which follows from

$$\sum_j p_j = P \bmod 2\pi. \tag{3.6}$$

The central idea of our method is to introduce an auxiliary sum replacing the constraint so that we can sum over the quantum numbers independently. We introduce the sum

$$\frac{1}{A} \sum_{\alpha=1}^{A} e^{i\alpha(\sum_j p_j - P)}, \tag{3.7}$$

where A is an integer depending on N and L. If the momentum constraint is satisfied, then the sum gives 1, whereas in all other cases

$$\sum_{\alpha=1}^{A} e^{i\alpha(\sum_j p_j - P)} = e^{i(\sum_j p_j - P)} \frac{e^{iA(\sum_j p_j - P)} - 1}{e^{i(\sum_j p_j - P)} - 1}. \tag{3.8}$$

We need to choose A so that the r.h.s. above is always zero. Using the solution (3.4)-(3.5) we obtain the necessary and sufficient conditions (assuming N is even)

$$\frac{A}{N+L} \in \mathbb{Z}, \qquad \frac{AN}{L(N+L)} \in \mathbb{Z} \qquad \frac{A}{2L} \in \mathbb{Z}. \tag{3.9}$$

If we assume to have a fixed particle density, then A has to scale linearly with N. For example if $n = N/L = 1$ then we can choose $A = 2N$ to satisfy the conditions. However, depending on the symmetries of the initial state in many cases A can be set to a fixed small number. This will be discussed at the end of this section.

Now we are in the position to sum over the a-variables independently:

$$\mathcal{L}(t) = \frac{Z(N)^2}{AL(L+N)^{N-1}} \sum_{J=1}^{L} \sum_{\alpha=1}^{A} \frac{1}{N!} \sum_{a_1, \ldots, a_N} e^{i\alpha(\sum_j p_j - P)} e^{-i\sum_j E(p_j)t} \left|\det G_j(a_k)\right|^2. \tag{3.10}$$

We expand the determinants as

$$\left|\det G_j(a_k)\right|^2 = \sum_{\sigma, \bar{\sigma}} (-1)^{\sigma} (-1)^{\bar{\sigma}} \prod_j G_{\sigma_j}(a_k) G^*_{\bar{\sigma}_j}(a_k). \tag{3.11}$$

Each a variable appears exactly twice in a factorized form, therefore the summations over them can be performed independently. By renaming the a-variables and reordering the factors in the product we obtain a single sum and a factor of $(N!)$. The resulting sum has the form of a determinant, we thus obtain

$$\mathcal{L}(t) = \frac{Z(N)^2}{AL(L+N)^{N-1}} \sum_{J=1}^{L} \sum_{\alpha=1}^{A} e^{-i\alpha P} \det M, \tag{3.12}$$

with

$$M_{jk} = \sum_{I=1}^{N+L} e^{i\alpha p_I} e^{-iE(p_I)t} G_j(a_I) G_k^*(a_I), \tag{3.13}$$

where

$$a_I = e^{ip_I}, \qquad p_I = I\frac{2\pi}{N+L} + \frac{\pi+P}{N+L}, \qquad P = J\frac{2\pi}{L}, \tag{3.14}$$

and the energy is given by $E(p_I) = 4\sin^2(p_I/2)$. We stress that in (3.5) there are N a-variables for each state and they correspond to the physical rapidities of the individual states, whereas in (3.14) the a-variables form a list of the $N+L$ possible particle momenta for each overall momentum P. To distinguish the two different roles of rapidities we use upper case indices for the non-physical rapidity variables.

Formula (3.12) expresses the Loschmidt echo as a sum of LA determinants, where the summation variables are the overall momentum quantum number J and the auxiliary parameter α . Each matrix entering this sum can be written as

$$M = \tilde{G}\Lambda\tilde{G}^\dagger, \tag{3.15}$$

where $\tilde{G}$ is an $N \times (L+N)$ matrix with elements given by

$$\tilde{G}_{jK} = G_j(a_K), \tag{3.16}$$

and Λ is an $(L+N) \times (L+N)$ diagonal matrix with elements

$$\Lambda_{IK} = \delta_{IK} e^{i\alpha p_K} e^{-iE(p_K)t}. \tag{3.17}$$

Note that $\tilde{G}$ only depends on J, whereas Λ depends on both α and J. Our derivation of (3.12)-(3.15) can be regarded as a backwards application of the Cauchy-Binet formula, which expresses the determinant of a product of non-square matrices as a sum over subsets of the rows.

Depending on the situation, formula (3.12) can be transformed into more convenient representations. One way to simplify it is to introduce the Fourier transform operator over N points:

$$V_{kl} = \frac{1}{\sqrt{N}} e^{-iq_k l} \qquad q_k = \frac{2\pi}{N}k + \frac{\pi+P}{N}. \tag{3.18}$$

For later convenience we introduced a shift in the q-variables so that

$$(e^{iq_k})^N = -e^{iP}, \qquad k = 1, \ldots, N. \tag{3.19}$$

It can be verified that V is a unitary operator. Inserting V and $V^\dagger$ into (3.15) the Loschmidt echo is written as

$$\mathcal{L}(t) = \frac{1}{AL(L+N)^{N-1}} \sum_{J=1}^{L} \sum_{\alpha=1}^{A} e^{-i\alpha P} \det\left(H\Lambda H^\dagger\right), \tag{3.20}$$

where

$$H_{jK} = \frac{1}{\sqrt{N}} \sum_m e^{-iq_j m} G_m(a_K). \tag{3.21}$$

This form is particularly useful if the initial state is a pure local Fock state with G given by (3.2), because in those cases we obtain a Fourier-like sum:

$$H_{jK} = \frac{1}{\sqrt{N}} \sum_m e^{-iq_j m} e^{ip_K(x_m+m)}. \tag{3.22}$$

If the initial state has periodic structure, then the matrix H can only have few non-vanishing matrix elements. This makes formula (3.20) very effective. Note that the q-variables form a set of N whereas the p-variables form a set of $N+L$, and they have different definitions (3.18) and (3.14).

Depending on the initial state, many of the determinants in (3.20) can be equal, thus the parameter A can be set to a fixed small number. Examples include periodic pure Fock states. In appendix B it is shown that if x_j are the initial positions of the particles in the Fock state and a new Fock state constructed from the coordinates $y_j = j + x_j$ has periodicity ν, then it is enough to choose $A = \nu$.

3.2 Exact results for the emptiness formation probability

Here we compute the time evolution of a simple physical observable, namely the m-site emptiness formation probability (EFP). The EFP has been intensively studied in the ground state of various integrable systems such as the XXZ chain [25, 26, 32, 33, 34, 35, 36, 37], free-fermion Hamiltonians [38, 39, 40] as well as the q-boson model [21, 22, 23, 24]. Recently, there has even been some progress on the time evolution of the EFP after a quench for free fermions [41].

We define local projection operators Π_j^0 that project to the subspace with no particle on site j. Then the position dependent m-site EFP is defined as

$$E_m(j) = \prod_{k=0}^{m-1} \Pi_{j+k}^0.$$

Inserting two complete sets of states the time evolution is computed as

$$\langle E_m(j,t)\rangle = \langle\Psi_0|\{a\}_N\rangle\ \langle\{a\}_N|E_m(j)|\{b\}_N\rangle\ \langle\{b\}_N|\Psi_0\rangle\ e^{-i(E_b-E_a)t}.$$

Matrix elements of the EFP operators were first derived in [21] using the ABA, but they can be calculated from the coordinate BA wave functions too [24]. In our normalization the matrix element between two un-normalized off-shell Bethe states reads

$$\langle\{a\}_N|E_m(j)|\{b\}_N\rangle = e^{i(P_2-P_1)j} \det\left[\frac{1-\left(\frac{b_l}{a_k}\right)^{N+L-m-1}}{1-\frac{b_l}{a_k}}\right], \tag{3.23}$$

where

$$e^{iP_1} = \prod_{j=1}^{N} a_j \qquad e^{iP_2} = \prod_{j=1}^{N} b_j.$$

If two rapidities coincide then the corresponding matrix element has to be evaluated using l'Hôpital's rule.

The matrix on the r.h.s. of (3.23) has the property that its lth row (or kth column) depends on b_l (or a_k), respectively. Together with the similar form of the overlap matrices, this makes it possible to perform essentially the same steps as in the previous section. The difference is that here we are faced with a double sum over intermediate states, leading to two auxiliary sums. This yields the following final result:

$$\langle E_m(j,t)\rangle = \frac{Z^2(N)}{(AL(L+N)^{N-1})^2} \sum_{J_1,J_2=1}^{L} e^{i(P_2-P_1)j} \sum_{\alpha_1,\alpha_2=1}^{A} e^{-i\alpha_1 P_1} e^{i\alpha_2 P_2} \det O_m. \tag{3.24}$$

Here $J_{1,2}$ are the momentum quantum numbers, $\alpha_{1,2}$ are the auxiliary variables, and the $N\times N$ matrix O_m is given by

$$O_m = \tilde{G}_1 \Lambda_1 F_m \Lambda_2 \tilde{G}_2^\dagger, \tag{3.25}$$

where $\tilde{G}_1$ and $\tilde{G}_2$ are $N\times(N+L)$, and $\Lambda_{1,2}$ and F_m are $(N+L)\times(N+L)$ matrices with elements

$$\tilde{G}_{1,lK} = G_l(a_K), \qquad \tilde{G}_{2,lK} = G_l(b_K), \tag{3.26}$$

$$\Lambda_{1,IK} = \delta_{IK} e^{i\alpha_1 p_K + iE(p_K)t} \qquad \Lambda_{2,IK} = \delta_{IK} e^{-i\alpha_2 q_K - iE(q_K)t}, \tag{3.27}$$

and finally

$$F_{m,IK} = \frac{1 - \left(\frac{b_I}{a_K}\right)^{N+L-m-1}}{1 - \frac{b_I}{a_K}}.$$

The parametrization of the a and b variables reads

$$\begin{aligned} a_I &= e^{ip_I}, \qquad p_I = I\frac{2\pi}{N+L} + \frac{\pi + P_1}{N+L}, \qquad P_1 = J_1\frac{2\pi}{L} \\ b_I &= e^{iq_I}, \qquad q_I = I\frac{2\pi}{N+L} + \frac{\pi + P_2}{N+L}, \qquad P_2 = J_2\frac{2\pi}{L}. \end{aligned} \tag{3.28}$$

In those cases where $J_1 \neq J_2$ we can use the Bethe equations to write the matrix F_m as

$$F_{m,IK} = \frac{1 - e^{i(P_2-P_1)}\left(\frac{a_K}{b_I}\right)^{m+1}}{1 - \frac{b_I}{a_K}}.$$

On the other hand, for $J_1 = J_2$ we obtain

$$F_{m,IK} = (N+L)\delta_{IK} - \frac{a_K}{b_I}\frac{1 - \left(\frac{a_K}{b_I}\right)^{m+1}}{1 - \frac{a_K}{b_I}}. \tag{3.29}$$

Similar to the computation of the Loschmidt echo, we can insert the matrices V defined in (3.18) into (3.24). This gives an alternative representation for the O-matrices:

$$O_m = H\Lambda_1 F_m \Lambda_2 H^\dagger, \tag{3.30}$$

with H given by (3.21).

Formula (3.24) is exact and it computes the EFP with $\mathcal{O}(L^2A^2(N+L)^2N) \sim \mathcal{O}(N^7)$ steps. Although the power 7 is quite large, this is still a huge simplification over the original exponential sums. Moreover, depending on the situation, many terms can be exactly identical which can lead to substantial improvements of the numerical algorithm.

If we restrict ourselves to calculate the space averaged EFP, it is enough the keep terms with $J_1 = J_2$:

$$\langle \bar{E}_m(t) \rangle \equiv \frac{1}{L}\sum_{j=1}^{L} \langle E_m(j,t) \rangle = \frac{Z^2(N)}{(AL(L+N)^{N-1})^2} \sum_{J=1}^{L} \sum_{\alpha_1,\alpha_2=1}^{A} e^{i(\alpha_2-\alpha_1)P} \det O_m, \tag{3.31}$$

where the matrix O_m is defined in (3.25) and it still depends on J and $\alpha_{1,2}$.

Further simplification is possible when the initial state is invariant under translations by $p \geq 1$ sites. In these cases the sum in (3.24) can be restricted to

$$J_{1,2} = \frac{L}{p}\tilde{J}_{1,2}, \qquad \tilde{J}_{1,2} = 1, \ldots, p.$$

In these cases the sum in (3.24) is more manageable because its computational cost is $\mathcal{O}(N^5)$ instead of the original $\mathcal{O}(N^7)$. If full translational invariance holds ($p = 1$) then only the zero momentum sectors contribute:

$$\langle E_m(t) \rangle = \frac{Z^2(N)}{(AL(L+N)^{N-1})^2} \sum_{\alpha_1,\alpha_2=1}^{A} \det O_m. \tag{3.32}$$

If the initial state is a pure Fock state, with partial or full translational invariance, then depending on its actual structure the number A can be set to a constant value, and the cost of the resulting numerical calculations becomes $\mathcal{O}(N^3)$. In appendix B we also show that in these cases (3.32) can be transformed into a form which is identical to that obtained from the mapping to the XX model introduced in the next section.

4 Mapping to the XX chain

The XX chain is a spin-1/2 model that can be mapped to free fermions. It is defined through the Hamiltonian

$$H_{\mathrm{XX}} = -\sum_{j=1}^{M}(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+ - h\sigma_j^z), \tag{4.1}$$

where σ_j^+, σ_j^-, σ_j^z are the Pauli matrices acting on site j.

In the following we describe two mappings between the phase model and two different fermionic models . The first one connects the phase model with a slightly modified version of the XX model; the modification consists of an extra non-local boundary term. The second mapping connects the zero-momentum sectors of the phase model and the original XX model. This mapping is exact, and there are no extra boundary terms. In both cases we will work within a subspace with fixed total particle number N. The length of the bosonic and fermionic chains is L and M, respectively.

As a first step we introduce the notations for the basis vectors. In the same way as in (2.1) the states will be labeled by listing the occupation numbers, but we add subscripts B (Bosons) and F (Fermions) to distinguish the states of the two systems from each other:

$$|\mathbf{n}\rangle_B = |n_1\rangle_{\mathrm{B}} \otimes |n_2\rangle_{\mathrm{B}} \otimes \cdots \otimes |n_L\rangle_{\mathrm{B}}, \qquad \sum_{j=1}^{L} n_j = N$$

$$|\mathbf{m}\rangle_F = |m_1\rangle_{\mathrm{F}} \otimes |m_2\rangle_{\mathrm{F}} \otimes \cdots \otimes |m_M\rangle_{\mathrm{F}}, \qquad \sum_{j=1}^{M} m_j = N.$$

In the fermionic case the allowed local Fock states are $|0\rangle_{\mathrm{F}}$ and $|1\rangle_{\mathrm{F}}$.

Now we define the mapping from the phase model to a modified XX model. The main idea of this mapping first appeared in [24], but the boundary conditions and the extra non-local boundary term were not described there.

As a first step we introduce a mapping P from a local bosonic space to an arbitrary product of fermionic spaces by the simple action

$$P|n\rangle_{\mathrm{B}} = \underbrace{|1\rangle_{\mathrm{F}} \otimes |1\rangle_{\mathrm{F}} \otimes \cdots \otimes |1\rangle_{\mathrm{F}}}_{\text{n times}} \otimes |0\rangle_F.$$

In particular $P|0\rangle_{\mathrm{B}} = |0\rangle_{\mathrm{F}}$.

The mapping can be extended to the full bosonic Hilbert space naturally:

$$P|\mathbf{n}\rangle_B = \otimes_{j=1}^{L} \left(P|n_j\rangle_{\mathrm{B}}\right).$$

Examples are

$$P|2,3,0,1\rangle_B = |1,1,0,1,1,1,0,0,1,0\rangle_F \qquad P|0,1,0\rangle_B = |0,1,0,0\rangle_F$$

Another way to establish the mapping is through the coordinates of the particles. If the bosonic particles have coordinates x_j, $j = 1,\ldots,N$, such that $x_j \geq x_k$ for $j > k$, then the positions on the fermionic chain are $y_j = x_j + j - 1$. Note that this definition of the mapping is suggested by the explicit wave function (2.5), which becomes a Slater determinant in the new coordinates.

Note that P conserves the total particle number N and it connects the bosonic chain with length L and the fermionic chain with length $M = L + N$. However, the last site of the fermionic chain is not dynamic, its state remains fixed to $|0\rangle_F$. Another way to formulate the mapping is to delete the last site of the fermionic chain and then we would get a bijection between the N-particle sector of the the phase model of length L and the N-particle sector of the fermionic chain with length $N + L - 1$. However, for future convenience we keep the last site as well.

Using the mapping P we pull back the q-boson Hamiltonian to the fermionic chain. We show that in the bulk we obtain the XX model, and at the boundary a new term arises. The

fermionic Hamiltonian will be established by fixing its matrix elements. In both the q-boson and XX models the Hamiltonian is a sum of the hopping terms and the particle number term. The mapping conserves number of particles, therefore the particle number operators correspond to each other if the magnetic field in the XX model is chosen to be $h = 2$. On the other hand, the hopping terms are more complicated.

A transition matrix element is non-zero in both models if and only if there is a one-site hopping of one particle. On the bosonic side we consider a hopping from site j to site $j+1$. Let

$$|a\rangle_B = |\alpha\rangle_{\mathrm{B}} \otimes |a_j, a_{j+1}\rangle_{\mathrm{B}} \otimes |\beta\rangle_{\mathrm{B}} \qquad |a'\rangle_B = |\alpha\rangle_{\mathrm{B}} \otimes |a_j - 1, a_{j+1} + 1\rangle_{\mathrm{B}} \otimes |\beta\rangle_{\mathrm{B}}.$$

Here $|\alpha\rangle_{\mathrm{B}}$ (or $|\beta\rangle_{\mathrm{B}}$) is a state of the segment of the chain from sites 1 to $j-1$ (or $j+2$ to L), respectively. The Hamiltonian has a transition matrix element

$${}_B\langle a|H_{\mathrm{B}}|a'\rangle_B = -1.$$

Then

$$|a\rangle_F = P|a\rangle_B = \big(P|\alpha\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{a_j-1 \text{ times}}, 1, 0, \underbrace{1,\ldots,1}_{a_{j+1} \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\beta\rangle_{\mathrm{B}}\big)$$

$$|a'\rangle_F = P|\tilde{a}\rangle_B = \big(P|\alpha\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{a_j-1 \text{ times}}, 0, 1, \underbrace{1,\ldots,1}_{a_{j+1} \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\beta\rangle_{\mathrm{B}}\big).$$

Therefore

$${}_F\langle a|H_{\mathrm{XX}}|a'\rangle_F = -1.$$

Note that even though the matrix elements are the same, the real location of the fermionic hopping depends on the particle content of the states $|\alpha\rangle_{\mathrm{B}}$ and $|\beta\rangle_{\mathrm{B}}$.

Particle hoppings between the first and last sites of the q-boson model need special treatment, and they result in new non-local terms in the fermionic Hamiltonian. Let us investigate the two vectors

$$|b\rangle_B = |b_1\rangle_{\mathrm{B}} \otimes |\gamma\rangle_{\mathrm{B}} \otimes |b_L\rangle_{\mathrm{B}} \qquad |b'\rangle_B = |b_1 - 1\rangle_{\mathrm{B}} \otimes |\gamma\rangle_{\mathrm{B}} \otimes |b_L + 1\rangle_{\mathrm{B}},$$

where $|\gamma\rangle_{\mathrm{B}}$ is an arbitrary vector of the chain segment between sites 2 and $L-1$. We have the matrix element

$${}_B\langle b|H_{\mathrm{B}}|b'\rangle_B = -1.$$

On the other hand, the mapping gives

$$|b\rangle_{\mathrm{F}} = P|b\rangle_{\mathrm{B}} = \Big|\underbrace{1,\ldots,1}_{b_1 \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\gamma\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{b_L \text{ times}}, 0\Big\rangle_{\mathrm{F}}$$

$$|b'\rangle_{\mathrm{F}} = P|b'\rangle_{\mathrm{B}} = \Big|\underbrace{1,\ldots,1}_{b_1-1 \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\gamma\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{b_L+1 \text{ times}}, 0\Big\rangle_{\mathrm{F}}.$$

Alternatively these two vectors can be written as

$$|b\rangle_{\mathrm{F}} = P|b\rangle_{\mathrm{B}} = U\left[\Big|\underbrace{1,\ldots,1}_{b_1-1 \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\gamma\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{b_L \text{ times}}, 0, 1\Big\rangle_{\mathrm{F}}\right]$$

$$|b'\rangle_{\mathrm{F}} = P|b'\rangle_{\mathrm{B}} = \Big|\underbrace{1,\ldots,1}_{b_1-1 \text{ times}}, 0\Big\rangle_{\mathrm{F}} \otimes \big(P|\gamma\rangle_{\mathrm{B}}\big) \otimes \Big|\underbrace{1,\ldots,1}_{b_L \text{ times}}, 1, 0\Big\rangle_{\mathrm{F}},$$

where U is the periodic shift operator by one site to the right. Therefore

$$ {}_{\rm F}\langle b|UH_{\rm XX}|b'\rangle_{\rm F} = -1. \tag{4.2}$$

and in the N-particle sector we have the similarity relation between the Hamiltonians with sites L and $M = N + L$ given by

$$ H_{\rm B} = P^{-1} H_{\rm XX}^{mod} P \tag{4.3}$$

with

$$ H_{\rm XX}^{mod} = -\left[\sum_{j=1}^{M-2} (\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) + U\sigma_{M-1}^-\sigma_M^+ + \sigma_{M-1}^+\sigma_M^- U^{-1}\right] + 2N. \tag{4.4}$$

Notice that $H_{\rm XX}^{mod}$ respects the constraint that the last site of the fermionic chain is fixed to $|0\rangle_F$.

Now we are in the position to establish a modified mapping $\tilde{P}$ which connects the zero-momentum sectors of the phase and XX models. The idea is to symmetrize the original operator P with respect to translations, both on the bosonic and the fermionic side. To define $\tilde{P}$ we first need to fix a basis in the zero-momentum sectors. This can be achieved by selecting vectors $|\tilde{a}\rangle_{B,F}$ in the form of

$$ |\tilde{a}\rangle_{B,F} = \frac{1}{\sqrt{p^a_{B,F}}}\left[|a\rangle_{B,F} + U|a\rangle_{B,F} + U^2|a\rangle_{B,F} + \cdots + U^{p^a_{B,F}-1}|a\rangle_{B,F}\right], \tag{4.5}$$

where $|a\rangle_{B,F}$ are Fock states of the bosonic and fermionic chains that are invariant under translation by $p^a_{B,F}$ sites. Typically $p^a_B = L$ and $p^a_F = N + L$, but we also need to treat the special cases with partial translational invariance. An example is the zero-momentum Néel state in the fermionic chain which has $p_F = 2$ and is of the form

$$ \left|\tilde{N}\right\rangle_F = \frac{1}{\sqrt{2}}\left[|1010\cdots\rangle_F + |0101\cdots\rangle_F\right]. $$

In both the bosonic and fermionic chains we can choose an arbitrary member in the sum (4.5) as the first (labelling) vector $|a\rangle_{B,F}$. However, in the fermionic case we require that the last site of $|a\rangle_F$ is in the state $|0\rangle_F$. This can always be achieved by a finite number of translations because the fully occupied fermionic state does not occur in the image of the original mapping P.

The mapping $\tilde{P}$ is constructed by associating

$$ |\tilde{a}\rangle_F = \tilde{P}|\tilde{a}\rangle_B $$

such that for the original vectors

$$ |a\rangle_F = P|a\rangle_B \quad \text{and} \quad p^a_F = \frac{N+L}{L} p^a_B. $$

The relation on the right above follows simply from the properties of the mapping P. It is easy to see that $\tilde{P}$ establishes a bijection between the zero-momentum basis vectors of the bosonic and fermionic chains. Now we show that pulling back the q-boson Hamiltonian leads to the XX model.

First of all notice, that a one-particle hopping amplitude between two vectors $|a\rangle_{B,F}$ and $|b\rangle_{B,F}$ is non-zero only in those cases when at least one of the states has $p^{a,b}_B = L$ or $p^{a,b}_F = N + L$, respectively. If neither states are periodic, then it can be seen immediately that the one-particle hopping amplitude is -1 both in the bosonic and fermionic chains. Now we consider the cases when $p^a_B < L$, $p^a_F < N + L$ and $p^a_B = L$, $p^a_F = N + L$. In these cases breaking of the partial translational invariance is a result of a single hopping somewhere along the chain. It is easy to see that

$$ {}_B\langle\tilde{a}\,|H_B|\,\tilde{b}\rangle_B = -\frac{L}{\sqrt{p^a_B p^b_B}} = -\sqrt{\frac{L}{p^a_B}} \quad \text{and} \quad {}_F\langle\tilde{a}|H_{XX}|\tilde{b}\rangle_F = -\frac{N+L}{\sqrt{p^a_F p^b_F}} = -\sqrt{\frac{N+L}{p^a_F}}. $$

Using the relation $p_F^a = \frac{N+L}{L} p_B^a$, we obtain that the two matrix elements are indeed the same. With this we have established that in the zero-momentum sectors of each theory

$$H_{\mathrm{B}} = \tilde{P}^{-1} H_{\mathrm{XX}} \tilde{P}. \tag{4.6}$$

It is useful to consider the physical S-matrix in the two models. In the XX model we have $S = -1$, whereas in the phase model S is given by (2.6). The apparent difference is easily explained by the non-locality of the mapping P. Two particles that are on neighboring sites of the phase model correspond to a string of 101 on the XX chain. Therefore, exchanging positions of two particles in the phase model is equivalent to performing a translation, an exchange and another translation in the XX model. Multiplying the phase factors associated to these three steps, we obtain the bosonic S-matrix (2.6).

4.1 Other local operators

The mappings between the models are highly non-local, therefore there is no general correspondence between local operators. However, we can find a limited set of operators that are local in both models and whose space averages correspond to each other. Examples include the emptiness formation probability (EFP) operators and the projection operators that measure the probability to have a fixed number of particles on a given site.

First we consider the EFP. We define local projection operators[3] $\Pi^0_{j,B}$ and $\Pi^0_{j,F}$ that project to the subspaces where there is no particle on site j. Then the position dependent m-site EFP is defined as

$$E_{m,B}(j) = \prod_{k=0}^{m-1} \Pi^0_{j+k,B}, \qquad E_{m,F}(j) = \prod_{k=0}^{m-1} \Pi^0_{j+k,F}.$$

The average EFP is

$$\bar{E}_{m,B} = \frac{1}{L} \sum_{j=1}^{L} E_{m,B}(j), \qquad \bar{E}_{m,F} = \frac{1}{M} \sum_{j=1}^{M} E_{m,F}(j).$$

It is easy to see that P maps a local bosonic m-site EFP to a local fermionic $m+1$ site EFP, but the position of the resulting fermionic operator depends on the particle distribution of the bosonic state outside the EFP. However, the averaged EFP's are mapped onto each other, except for finite boundary contributions:

$$\bar{E}_{m,B} \approx \frac{M}{L} P^{-1} \bar{E}_{m+1,F} P = (1+n) P^{-1} \bar{E}_{m+1,F} P, \tag{4.7}$$

where the factor $\frac{M}{L} = (1+n)$ arises from the change of the volume. Equality is reached in the thermodynamic limit. Regarding the mapping $\tilde{P}$ connecting the zero-momentum sectors it is easy to see that the relation

$$\bar{E}_{m,B} = (1+n) \tilde{P}^{-1} \bar{E}_{m+1,F} \tilde{P} \tag{4.8}$$

is exact both in finite volume and in the thermodynamic limit.

As a second example we consider the probability to have m particles on site j of the bosonic model. To this end we introduce the general projection operators $\Pi^m_{j,B}$ and $\Pi^m_{j,F}$, where $m = 0 \ldots \infty$ and $m = 0, 1$ in the bosonic and fermionic models, respectively. We will also use the average of the bosonic operator

$$\bar{\Pi}^m_B = \frac{1}{L} \sum_{k=1}^{L} \Pi^m_{j,B}.$$

[3]The bosonic EFP operator has already been defined in Section 3.2. Here we redefine it with a subscript B in order to have the same notations for the bosonic and fermionic operators.

A bosonic site with m particles is always mapped to a continuous sequence of m particles on the fermionic chain, preceded and followed by an empty site:

$$P|\dots,m,\dots\rangle_{\mathrm{B}} = \Big|\dots,0,\underbrace{1,\dots,1}_{m\ \text{times}},0,\dots\Big\rangle_{\mathrm{F}}\,.$$

We define the fermionic operators and their average

$$D_{m,F}(j) = \Pi^0_{j-1,F}\left(\prod_{k=0}^{m-1}\Pi^1_{j+k,F}\right)\Pi^0_{j+m,F}, \qquad \bar{D}_{m,F} = \frac{1}{M}\sum_{j=1}^{M} D_{m,F}(j).$$

Once again, the averaged operators correspond to each other except for certain finite boundary terms:

$$\bar{\Pi}^m_B \approx (1+n)P^{-1}\bar{D}_{m,F}P. \tag{4.9}$$

With regard to the mapping $\tilde{P}$ in the zero-momentum sector we have an exact relation

$$\bar{\Pi}^m_B = (1+n)\tilde{P}^{-1}\bar{D}_{m,F}\tilde{P}. \tag{4.10}$$

Note that (4.9)-(4.10) give the EFP relations (4.7)-(4.8) at the special value $m = 0$.

4.2 Formation probabilities in the XX chain

As discussed in the previous subsection, the calculation of averaged formation probabilities in the bosonic case can be directly recast in the language of the XX chain with Hamiltonian (4.1). The correspondence holds also for the time dependent quantities after a quench and becomes especially useful in case of the translational invariant initial states. Here we use this mapping to obtain simple determinant formulas for the EFP, employing free-fermion techniques [38, 39]. On the one hand, these will serve as a check of the bosonic results, but they will also be used to obtain new results. For simplicity, we will only consider cases when the bosonic state is translationally invariant.

The periodic XX chain can be transformed into a fermionic hopping model with Hamiltonian

$$H_F = -\sum_{j=1}^{M}(c^\dagger_{j+1}c_j + c^\dagger_j c_{j+1} - h(2c^\dagger_j c_j - 1)), \tag{4.11}$$

where, in the sector of even particle number $N = \sum_j c^\dagger_j c_j$, the boundary condition is anti-periodic in the fermion operators, $c_{M+1} = -c_1$. Note that the lengths M and L of the fermionic and bosonic chains are related as $M = L + N$. Since the term proportional to h in (4.11) is just a constant, it drops out from the time evolution and one can set $h = 0$. The Hamiltonian becomes diagonal after a Fourier transform

$$H_F = -\sum_{k=1}^{M} 2\cos(q_k)c^\dagger_{q_k}c_{q_k}\,, \qquad q_k = \frac{\pi}{M}\,(2k-1)\,, \tag{4.12}$$

where the allowed values of momenta are set by the anti-periodic boundary condition.

Starting from an initial Fock state, all the information about the time evolved state is encoded in the fermionic correlation matrix elements $C_{mn}(t) = \langle c^\dagger_m(t)c_n(t)\rangle$. Using the diagonal form of H_F, the time evolution of the fermionic operators reads

$$c_n(t) = \sum_l U_{nl}(t)c_l\,, \quad U_{nl}(t) = \frac{1}{M}\sum_k e^{-iq_k(n-l)}e^{i2t\cos q_k}. \tag{4.13}$$

From here on, we will work directly in the thermodynamic limit $M \to \infty$, where the matrix elements of the unitary time evolution operator are given via Bessel functions as $U_{nl}(t) = i^{n-l}J_{n-l}(2t)$. The time dependent correlation matrix elements then follow as [42]

$$C_{mn}(t) = i^{n-m}\sum_{k,l} i^{k-l}J_{m-k}(2t)J_{n-l}(2t)C_{kl}(0)\,. \tag{4.14}$$

The knowledge of the above two-point functions allows us to calculate the expectation value of arbitrary products of creation and annihilation operators through Wick's theorem. In particular, the position dependent m-site EFP is given by [38, 39]

$$\langle E_{m,F}(j,t)\rangle = \langle \prod_{k=0}^{m-1} c_{j+k}(t)c^{\dagger}_{j+k}(t)\rangle = \det(\mathbb{1} - C_{I_{j,m}}(t)), \tag{4.15}$$

where $I_{j,m} = [j, j+m-1]$ and for any interval I the matrix C_I is the reduced correlation matrix with both its rows and columns restricted to I. Finally, the averaged bosonic EFP is obtained as

$$\left\langle \bar{E}_{m,B}(j,t)\right\rangle = \frac{1}{L}\sum_{j=1}^{N+L} \det(\mathbb{1} - C_{I_{j,m}}(t)). \tag{4.16}$$

If the bosonic initial state is such that its fermionic picture is invariant under translations under ν sites then we have the simpler formula

$$\left\langle \bar{E}_{m,B}(j,t)\right\rangle = \frac{1+n}{\nu}\sum_{j=1}^{\nu} \det(\mathbb{1} - C_{I_{j,m}}(t)). \tag{4.17}$$

For the sake of completeness, we show in Appendix B that (4.17) can be derived directly from formula (3.32) without invoking the mapping to the XX model.

One can also obtain determinant formulas for the various fermionic string formation probabilities $\langle D_{m,F}(j,t)\rangle$ that are related via (4.9) to the probability of finding m bosons at a given site. In particular, for $m = 1$ one has

$$\langle D_{1,F}(j,t)\rangle = \det(\mathbb{1} - C_{I_1}) - \det(\mathbb{1} - C_{I_2}), \tag{4.18}$$

where $I_1 = [j-1] \cup [j+1]$ and $I_2 = [j-1, j+1]$. Note, that this is just the EFP on two next-nearest-neighbor sites $j-1$ and $j+1$ minus the EFP on all three consecutive sites, which indeed gives the probability of finding a 010 string.

Finally, we consider the string-probability $D_{2,F}(j,t)$ on sites $[j-1, j+2]$ in the time evolved state. This can again be written in terms of EFP's as

$$\langle D_{2,F}(j,t)\rangle = \det(\mathbb{1} - C_{I_1}) - \det(\mathbb{1} - C_{I_2}) - \det(\mathbb{1} - C_{I_3}) + \det(\mathbb{1} - C_{I_4}) \tag{4.19}$$

on the corresponding subsets

$$I_1 = [j-1] \cup [j+2]\,, \qquad I_2 = [j-1, j] \cup [j+2]\,,$$
$$I_3 = [j-1] \cup [j+1, j+2]\,, \qquad I_4 = [j-1, j+2]\,.$$

Formulas (4.18)-(4.19) will be used in the next section to derive the bosonic occupation probabilities in a specific quench situation.

5 Translationally invariant initial states

In this section we calculate exact time evolution starting from translationally invariant initial states. We deal with homogeneous initial states:

$$|\psi_n\rangle_B \equiv |n, n, n, n, \cdots\rangle_B$$

On the one hand these are natural candidates, because they could be prepared in experiments or other numerical simulations. On the other hand they are also convenient for our purposes because their overlaps have the form (3.1) so that the calculations of the previous sections can be applied.

Our methods enable us to investigate both finite size effects and the infinite volume limit. Concerning finite size effects, we can address the question of particle propagation on the background induced by the quench. In a finite volume global quench we expect that the physical observables will be very close to their infinite volume limit until a time $t \approx L/2v^*$, where v^* is interpreted as the velocity of propagation through the highly excited stationary

state reached after the quench. The physical interpretation can be given using the semi-classical reasoning of [43]: as a result of the quench quasi-particles are emitted at each site, and finite size effects arise only at times when two particles have traveled around the volume and meet at the opposite side. Our numerical data enables us to give estimates for the velocities v^*. We stress however, that these can not be considered as rigorous answers, because there are no sharp wave fronts associated with the finite size effects.

Concerning the infinite volume and infinite time limit, we can check the predictions of the GGE. In [27] it was argued that in the q-boson model the GGE should give a correct description of the stationary state after the quench[4]. The main argument was that in the q-boson model the initial values of the conserved charges completely determine what types of states can populate the system after the quench, in particular there is a one-to-one correspondence between the charges and the Bethe root densities. After dephasing, all local operators approach the averages in these Bethe states, and the averages only depend on the root densities. Therefore, the charges in the initial state completely determine the long-time limit of local observables. This was explicitly checked in [27] for the particular quench that will be discussed in 5.1. In the present work we can go further and consider more general initial states.

In [27] it was shown that the mean values of the EFP operators in Bethe states can be expressed using the charges, for example

$$\langle\Omega|E_1|\Omega\rangle = \frac{1}{1+n}\left(1-|\langle\Omega|Q_1|\Omega\rangle|^2\right),$$

where $|\Omega\rangle$ is an arbitrary on-shell Bethe state. It was also shown that in pure Fock states the expectation values of the charges are all zero except for the particle number operator. Therefore it is very easy to obtain the GGE predictions for the limit of the EFP, for example

$$\lim_{t\to\infty}\langle E_1(t)\rangle = \frac{1}{1+n}, \tag{5.1}$$

which should be valid for any quench starting from an initial state which is a pure Fock state. In particular, we will verify this for the states $|\psi_n\rangle_B$ at the end of this section.

5.1 Initial state $|\psi_1\rangle_B = |1,1,1,1\ldots\rangle_B$

This is probably the simplest initial state with particle density $n = N/L = 1$: there is exactly one particle at each site. The initial state corresponds to the particular component of the Bethe vector (2.5) with $x_j = j$, therefore the overlap can be written as

$$\langle\Psi_0|\{a\}_N\rangle = \det_N\left((a_k)^{2j}\right) = \prod_j a_j^2\prod_{j<k}(a_k^2-a_j^2) = \prod_{j<k}(a_k^2-a_j^2), \tag{5.2}$$

where we used that the overlap is non-zero for zero-momentum states only. This simple structure of the overlap made possible the direct computation of the time-evolution of the 1-EFP in [27]. The following result was obtained in finite volume:

$$\langle E_1(t)\rangle = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{N}\sum_j \cos(4\cos(p_j)t)\right) - \frac{1}{2}\left|\frac{1}{N}\sum_j \sin(4\cos(p_j)t)e^{ip_j}\right|^2, \tag{5.3}$$

where

$$p_j = \frac{\pi(2j-1)}{2L}. \tag{5.4}$$

Taking the infinite volume limit with fixed t leads to

$$\langle E_1(t)\rangle = \frac{1}{2} - \frac{1}{2}\left(\int_0^\pi \frac{dp}{\pi}\cos(4\cos(p)t)\right)^2 - \frac{1}{2}\left|\int_0^\pi \frac{dp}{\pi}\sin(4\cos(p)t)e^{ip}\right|^2. \tag{5.5}$$

[4]It might be more accurate to use the term "predictions of the Diagonal Ensemble (DE)", because an actual Generalized Gibbs Ensemble is not constructed. However, for historical reasons and to conform with the terminology of [10], we continue to use the expression "predictions of the GGE".

The GGE prediction (5.1) gives

$$\lim_{t\to\infty}\langle E_1(x,t)\rangle = \frac{1}{2},$$

and this is clearly confirmed by (5.5), as it was already observed in [27].

Quite interestingly the symmetrized mapping to the XX model gives

$$\tilde{P}|1,1,\ldots,1\rangle_{\rm B} = \frac{1}{\sqrt{2}}\left[|1,0,1,0,\cdots\rangle_{\rm F} + |0,1,0,1,\cdots\rangle_{\rm F}\right]. \tag{5.6}$$

Therefore, this particular situation corresponds to the Néel-to-XX quench studied extensively in [44, 45, 46, 47], given that the initial state is chosen to be the translationally invariant combination of the Néel and Anti-Néel states. These papers concentrated on the time evolution of the staggered anti-ferromagnetic order, the spin-spin correlations, and the Loschmidt echo.

Below we also show how to compute (5.3) in the fermionic language introduced in section 4. However, before turning to the XX model we first consider the Loschmidt amplitude. In appendix A.1 we compute it using the general formalism introduced in section 3. Our finite volume result for the amplitude reads

$$\mathcal{L}(t) = \left[\prod_{j=1}^{N}\cos(2\cos(p_j)t) + \prod_{j=1}^{N}\sin(2\cos(p_j)t)\right], \tag{5.7}$$

where p_j is defined in (5.4). This agrees with the corresponding result of [47]. The second term in (5.7) is not present in the Néel to XX quench, because it corresponds to the finite volume Néel to Anti-Néel transition. If we fix t then in the $L\to\infty$ limit only the first term survives. The physical reason for this is that in infinite volume the time evolution generated by the Hamiltonian can not shift the whole chain by one site in finite time. Therefore, in the TDL limit at finite t we have

$$\lim_{L\to\infty}\frac{1}{L}\log|\mathcal{L}(t)| = \frac{1}{\pi}\int_0^\pi \log|\cos(2\cos(p)t)|dp. \tag{5.8}$$

We note that even though the finite volume amplitude is exactly equal to that computed in [46], there is a factor of 2 in the Loschmidt echo per site. This factor arises from the re-scaling of the volume through the mapping between the two models.

Now we turn to the evaluation of the EFP and the local occupation probabilities. Formula (5.3) could be derived easily from (3.32) too, for example by using (3.30) for a simple evaluation of the determinants. However, it is instructive to re-derive it in the fermionic language introduced in section 4.

We are interested in the 2-site EFP $\langle E_{2,F}(j,t)\rangle$, which can be obtained via the time-dependent fermionic correlation matrix (4.14) using the determinant formula (4.15). The correlation matrix at time $t=0$ is diagonal with alternating entries $C_{2k-1,2k-1}(0)=1$ and $C_{2k,2k}(0)=0$. Substituting into (4.14) and using the Bessel function addition theorems [48]

$$\sum_{k=-\infty}^{\infty} J_{k+m}(z)J_{k+n}(z) = \delta_{m,n}, \qquad \sum_{k=-\infty}^{\infty}(-1)^k J_{k+m}(z)J_{k+n}(z) = (-1)^m J_{n-m}(2z), \tag{5.9}$$

one obtains the following simple expression for the time evolved matrix elements

$$C_{m,n}(t) = \frac{i^{n-m}}{2}\left[\delta_{m,n} - (-1)^m J_{n-m}(4t)\right]. \tag{5.10}$$

Although the matrix elements are invariant only to 2-site translations, it is easy to see that the 2×2 determinants defining the 2-site EFP in (4.15) are completely translationally invariant. Therefore, one has for the average EFP

$$\left\langle \bar{E}_{2,F}(t)\right\rangle = \frac{1}{4}(1 - J_0^2(4t) - J_1^2(4t)). \tag{5.11}$$

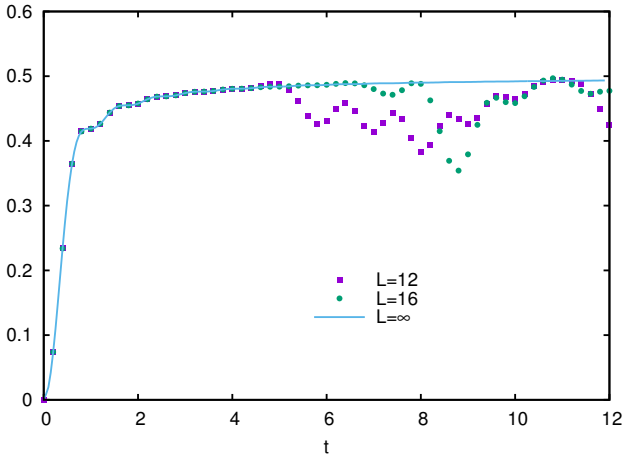

Figure 1: Time evolution of the 1-site EFP in the phase model for the initial state $|\psi_1\rangle_B = |1,1,1,1,\cdots\rangle_B$.

Comparing to the bosonic result in (5.5), one obtains the relation (4.7) with the volume re-scaling factor $1+n=2$. Moreover, as discussed in the previous section, one has an exact match even with the finite volume result (5.3), which is easily obtained from (5.11) by replacing the Bessel functions in (5.11) with the finite-M propagators in (4.13) and setting $M=2L$.

In Fig. 1 we plotted the numerical data for the bosonic 1-EFP. We can see that the finite volume time evolution follows the infinite volume curve until a time $t^*=L/2v^*$ which grows linearly with the volume. The speed v^* of the propagation could be estimated from the difference between the finite and infinite volume results, or simply by looking at the numerical data for large volumes. We choose the latter method and read off the approximate value $v^*\approx 1$. Note that the maximum speed of particles in vacuum was $v_{\max}=2$, which is the group velocity as calculated from the energy-momentum relation (2.8). This slowing down with a factor of 2 can be explained by the properties of the mappings P and $\tilde{P}$ between the bosonic and fermionic models. In the XX chain information always propagates with a maximal speed of $v=2$, regardless of the local spin content. This follows from the fact that the model is effectively free. On the other hand, the mappings are non-local and they involve a re-scaling of the volume whose extent depends on the local particle density. In fact, a segment of the bosonic chain with length l and density n is mapped to a segment of the XX model with length $l(1+n)$. Therefore, information in the bosonic chain can not propagate faster than $v^*(n)=2/(1+n)$. In the present case this leads to $v^*=1$, in agreement with the numerical data.

We continue by calculating the expectation value of the average string-probability $\langle \bar{D}_{1,F}(t)\rangle$ which, in the bosonic chain, corresponds to the probability of having a single boson at a site. We stress that this is a new result, which could not have been obtained using the formalism of Section 3, because the matrix elements of the corresponding operators are not known. The fermionic 010 string probability is position dependent, however, due to 2-site translational invariance, it is enough to consider $j=1$ and $j=2$. Using the matrix elements (5.10) and expanding the determinants in (4.18), one finds for the average string probability

$$\langle \bar{D}_{1,F}(t)\rangle = \frac{1}{2}\sum_{j=1,2}\langle D_{1,F}(j,t)\rangle = \frac{1}{8}(1+3J_0^2+2J_1^2-J_2^2), \tag{5.12}$$

where for brevity the arguments $(4t)$ of the Bessel functions were suppressed. For the bosonic occupation probability we thus obtain

$$\langle \Pi_B^1(t)\rangle = \frac{1}{4}(1+3J_0^2+2J_1^2-J_2^2). \tag{5.13}$$

Similarly, one can also calculate the 0110-string probability $\langle D_{2,F}(j,t)\rangle$, which is, in fact, translational invariant due to the symmetry of the problem and thus one has to consider $j=1$ only. Evaluating the determinants in (4.19) is straightforward but rather tedious.

After re-scaling one arrives at the lengthy formula

$$\langle \Pi_B^2(t) \rangle = \frac{1}{8} \left[(1-J_0^2)(1-J_0^2-3J_1^2-J_3^2) - 2(1+J_0^2)J_2^2 + 4J_0J_1J_2(J_1-J_3) \right.$$
$$\left. +2J_1J_3(J_1^2-J_2^2) + J_1^2J_3^2 + (J_1^2+J_2^2)^2 + 4(J_1^2+J_2^2) \right]. \tag{5.14}$$

The bosonic occupation probabilities $\langle \Pi_B^m(t) \rangle$ are shown in Fig. 2 for $m = 0, 1, 2$. One observes that the curves relax rapidly to their stationary values, given by $1/2, 1/4, 1/8$, respectively. This suggests that the number of bosons has a geometric distribution in the stationary state. In the fermionic picture it is easy to verify that this is indeed the case. In fact, the reduced density matrix ρ_I of an arbitrary finite interval I in the stationary state is given via the reduced correlation matrix C_I as [49]

$$\rho_I \propto \exp\Big(\sum_{i,j\in I} H_{i,j} c_i^\dagger c_j\Big), \qquad H = \ln \frac{1-C_I}{C_I}, \tag{5.15}$$

where $C_I = \lim_{t\to\infty} C_I(t)$. From Eq. (5.10) one has immediately $C_I = \mathbb{1}/2$ and hence the local steady state is a Gibbs state at infinite temperature. In turn, this implies that all string configurations have equal probabilities and thus

$$\lim_{t\to\infty} \langle \bar{D}_{m,F}(t) \rangle = 2^{-(m+2)}.$$

We stress that equations (5.13)-(5.14) are new results of this work.

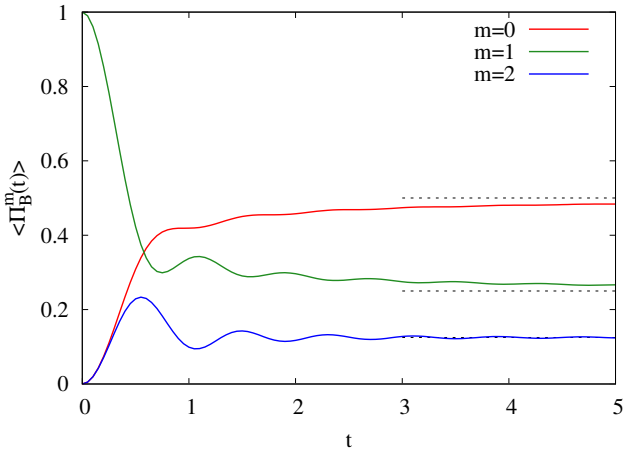

Figure 2: Time evolution of the probability to have m bosons on a single site for the initial state $|\psi_1\rangle_B = |1,1,1,1,\cdots\rangle_B$. The bosonic result is obtained from the averaged fermionic string probabilities $\langle \bar{D}_{m,F}(t) \rangle$ for $m = 0, 1, 2$ multiplied by the factor $1 + n = 2$, see Eq. (4.10). The horizontal dashed lines correspond to the $t \to \infty$ limit.

5.2 Initial state $|\psi_2\rangle_B = |2,2,2,2\ldots\rangle_B$

As a second example, we consider the initial state with exactly 2 particles at each site. The overlaps are of the form (3.1) with

$$G_j(a_k) = a_k^{j+x_j}, \text{ with } \quad x_j = \left[\frac{j+1}{2}\right],$$

and $[x]$ denotes the integer part. In this case the time evolution of the EFP can be computed either from formula (3.32) or by the mapping to the XX chain. In the following we present the latter computation.

The symmetrized mapping gives

$$\tilde{P}|2,2,\ldots\rangle_{\rm B} = \frac{1}{\sqrt{3}} \left[|1,1,0,1,1,0,\cdots\rangle_{\rm F} + |0,1,1,0,1,1,\cdots\rangle_{\rm F} + |1,0,1,1,0,1,\cdots\rangle_{\rm F} \right] \tag{5.16}$$

For simplicity we will work in the thermodynamic limit. Since the initial density has now a 3-site periodicity, we need a more general addition theorem of Bessel functions [48]

$$\begin{aligned}\sum_{k=-\infty}^{\infty} J_{n+k}(z)J_k(z)\cos(k\alpha) &= J_n(2\sin(\alpha/2)z)\cos(n(\pi-\alpha)/2)\\ \sum_{k=-\infty}^{\infty} J_{n+k}(z)J_k(z)\sin(k\alpha) &= J_n(2\sin(\alpha/2)z)\sin(n(\pi-\alpha)/2).\end{aligned} \tag{5.17}$$

Indeed, setting $\alpha = 2\pi/3$, we obtain two equations involving the various 3-periodic sums of products of Bessel functions. Supplementing the set of equations with the completeness relation on the l.h.s of Eq. (5.9), the system can be solved and the result can be used to evaluate the matrix elements in (4.14). In particular, the diagonal elements are obtained by choosing $n = 0$ in (5.17) with the result

$$C_{m,m}(t) = \begin{cases} \frac{1}{3}(2 + J_0(2\sqrt{3}t)) & \text{if } m = 3l+1 \text{ or } m = 3l+2 \\ \frac{2}{3}(1 - J_0(2\sqrt{3}t)) & \text{if } m = 3l. \end{cases} . \tag{5.18}$$

Similarly, the elements in the first off-diagonal follow from solving the set of equations for $n = 1$ and read

$$C_{m,m+1}(t) = \begin{cases} 0 & \text{if } m = 3l+1 \\ iJ_1(2\sqrt{3}t)/\sqrt{3} & \text{if } m = 3l+2 \\ -iJ_1(2\sqrt{3}t)/\sqrt{3} & \text{if } m = 3l. \end{cases} \tag{5.19}$$

Evaluating the determinants and taking the average, one arrives at

$$\langle \bar{E}_{2,F}(t)\rangle = \frac{1}{3}\sum_{j=1}^{3}\langle E_{2,F}(j,t)\rangle = \frac{1}{9}\left[1 - J_0^2(2\sqrt{3}t) - 2J_1^2(2\sqrt{3}t)\right]. \tag{5.20}$$

For the bosonic 1-EFP we thus obtain

$$\langle E_{1,B}(t)\rangle = \frac{1}{3}\left[1 - J_0^2(2\sqrt{3}t) - 2J_1^2(2\sqrt{3}t)\right]. \tag{5.21}$$

The finite volume result has the same form, with the Bessel functions replaced by the finite volume propagators (4.13).

In Fig. 3 we plot the numerical data for the 1-EFP in finite and infinite volume. Concerning the maximal speed of wave propagation, we can read off the approximate value $v^* \approx 2/3$. Similar to our previous example, the slowing down of the wave propagation is consistent with the re-scaling of the volume caused by $\tilde{P}$ with a factor of 3.

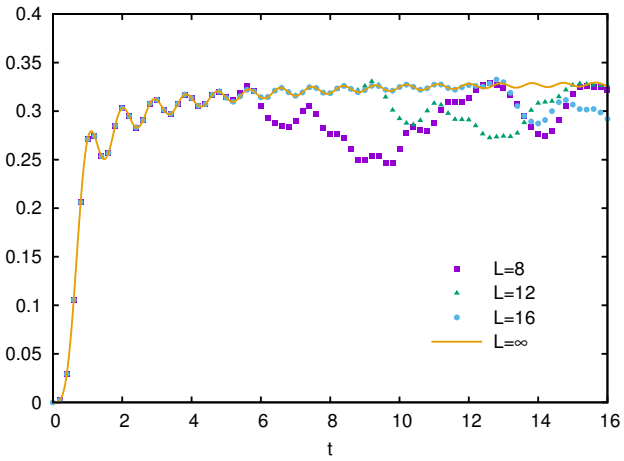

Figure 3: Time evolution of the 1-site EFP in the phase model for the initial state $|\psi_2\rangle_B = |2,2,2,2,\cdots\rangle_B$.

Note that the long-time limit of the 1-EFP in (5.21) is given by 1/3 which agrees with the GGE prediction (5.1). In fact, the prediction can even be verified for initial states $|\psi_n\rangle_B$

with arbitrary n. As observed already for $n = 1$ and $n = 2$, it is not difficult to prove in general that for $t \to \infty$ the off-diagonal matrix elements of the fermionic correlation matrix vanish, while the diagonal elements give the average fermionic density

$$\lim_{t\to\infty} C_{m,m}(t) = \frac{n}{n+1}, \qquad \lim_{t\to\infty} C_{m,m+1}(t) = 0. \tag{5.22}$$

Substituting into (4.17), one immediately finds

$$\lim_{t\to\infty} \langle E_{1,B}(t)\rangle = \frac{1}{1+n}. \tag{5.23}$$

In Appendix A.2 we also computed the time evolution of the Loschmidt amplitude, both in finite and infinite volume. The formulas are lengthy and we refrain from repeating them here. However, we note that the intermediate result (3.20) leads to a relatively simple calculation even in this case. In fact, it seems that regarding the Loschmidt amplitude the bosonic formulation always leads to simpler derivations than the fermionic methods.

6 Conclusions

In this work we investigated time evolution of physical observables in the phase model, which is the $q \to \infty$ limit of the q-boson model. We set up a framework to calculate the finite volume Loschmidt echo and the emptiness formation probability in non-equilibrium situations. One of our main results is equation (3.24), which computes the exact EFP in polynomial ($\mathcal{O}(N^7)$) time, which is a tremendous simplification as compared to the exponential sum over all states in the Hilbert space. Analogously, the Loschmidt echo can be obtained with $\mathcal{O}(N^5)$ steps. The result (3.24) applies whenever the overlaps with the initial state take the form (3.1). Pure Fock states in the local coordinate basis satisfy this requirement. Such states are natural candidates to study because they could be prepared in experiments or other independent numerical simulations.

The fact that the EFP could be obtained with a cost of $\mathcal{O}(N^7)$ steps shows that the complexity of the phase model is between that of a free theory and a generic Bethe Ansatz solvable model. In technical terms, this unique situation arises because the scattering phase shift (2.6) can be factorized. This leads to a special type of Bethe equation (2.7), which implies that the one-particle rapidities can be chosen freely from an enlarged set of $N + L$ solutions, as long as they obey an overall constraint coming from the total momentum quantization. Unfortunately, this simplification only emerges in the $q \to \infty$ limit, which restricts the scope of our method.

We also considered a non-local mapping between the phase model and the XX chain. In its simplest form, the mapping connects the zero-momentum sectors of the two theories, such that the Hamiltonians and space averages of certain other local operators are mapped onto each other. In situations that are not translationally invariant the mapping can not be used to give information about the space dependence of the observables, it applies only to the averaged operators. Therefore it is important to stress that the phase model should not be considered as "equivalent to a free theory".

In Section 5 we considered examples where the initial state is translationally invariant. We derived new results for the Loschmidt amplitude, the EFP, and local bosonic occupation probabilities using both the bosonic methods of section 3 and the mapping to the XX chain. In these formulas both the thermodynamic limit and the long time limit could be performed quite easily; in those cases where a GGE prediction was available for the stationary values in the TDL, the prediction was confirmed. We stress that, even though our results pertain to very special situations, they are among the very few that exactly solve time evolution and rigorously prove the validity of the GGE predictions in an interacting many-body system.

In a future publication we plan to apply our result (3.24) in situations that are not translationally invariant. Examples include global quenches starting from initial states that have periodic structure, for example $|\Psi_0\rangle_{\mathrm{B}} = |1, 0, 1, 0, \cdots\rangle_{\mathrm{B}}$. Formula (3.24) can be readily applied in those cases, giving the numerically exact finite volume EFP. However, it is a challenging open problem to obtain the thermodynamic limit of the space dependent EFP in such cases.

Another class of problems to be investigated is that of joining two subsystems with different initial states. This can be achieved for example by choosing initial states of the form $|\Psi_0\rangle_{\rm B} = |1,1,\cdots,1,2,2,\cdots,2\rangle_{\rm B}$. Our formulas deal with periodic boundary conditions, therefore this situation corresponds to having two initial domain walls on a circle. Time evolution will induce a particle and energy current at the two boundaries, and we plan to investigate the properties of these currents and the wave-front propagation into the bulk of the two subsystems.

Acknowledgments

The authors are grateful to Zoltán Zimborás for suggesting to study these problems, to Gábor Takács for helping to construct the mapping $\tilde{P}$ introduced in Section 4, to Michael Brockmann for discussions about the relation to the XX model and in particular for drawing our attention to Fig. 4 of [24] which describes the mapping P, and finally to Márton Kormos for very useful discussions which inspired us to construct the finite volume techniques of Section 3.

V.E. acknowledges funding from the Austrian Science Fund (FWF) through Lise Meitner Project No. M1854-N36.

A Analytic results for the Loschmidt amplitude

Here we compute exact analytic results for the Loschmidt amplitude using the formula (3.20). For simplicity we subtract an irrelevant constant from the Hamiltonian and will work with the one-particle dispersion relation

$$E(p) = -2\cos(p).$$

This only affects the phase of the Loschmidt amplitude.

A.1 Initial state $|\psi_1\rangle_B = |1,1,1,1,\dots\rangle_B$

We have $N = L$, the state is translationally invariant and only the $J = 0$ sector contributes. The initial coordinates of the particles are $x_j = j$. Then the matrix H is $N \times 2N$ and

$$H_{jK} = \frac{1}{\sqrt{N}} \sum_m e^{-iq_j m} e^{i2p_K m}. \tag{A.1}$$

We will use the definitions

$$q_j = \frac{2\pi}{N} j - \frac{\pi}{N}, \quad j = 1 \dots N \quad \text{and} \quad p_K = \frac{\pi}{N} K - \frac{\pi}{2N}, \quad K = 1 \dots 2N. \tag{A.2}$$

This gives

$$H_{jK} = \sqrt{N}(\delta_{j,K} + \delta_{j+N,K}). \tag{A.3}$$

This way the matrix product in (3.20) gives

$$(H\Lambda H^\dagger)_{jk} = N\delta_{j,k}\left(\Lambda_{j,j} + \Lambda_{j+N,j+N}\right) = N\delta_{jk}\left(e^{-iE(q_j/2)t}e^{i\alpha q_j/2} + e^{-iE(q_j/2+\pi)t}e^{i\alpha(q_j/2+\pi)}\right).$$

We have

$$\det(H\Lambda H^\dagger) = N^N \begin{cases} \prod_{j=1}^N 2\cos(2\cos(q_j/2)t)e^{i\alpha q_j/2} & \text{for even } \alpha \\ \prod_{j=1}^N 2i\sin(2\cos(q_j/2)t)e^{i\alpha q_j/2} & \text{for odd } \alpha \end{cases}.$$

For the sum of the momenta we obtain

$$\sum_{j=1}^N q_j = -\pi + \frac{2\pi}{N}\frac{N(N+1)}{2} = \pi N. \tag{A.4}$$

Therefore

$$\prod_{j=1}^{N} e^{i\alpha q_j/2} = e^{i\alpha\pi N/2} = \begin{cases} 1 & \text{for even } \alpha \\ (-1)^{N/2} & \text{for odd } \alpha \end{cases}. \tag{A.5}$$

This gives

$$\det(H\Lambda H^\dagger) = 2^N N^N \begin{cases} \prod_{j=1}^{N} \cos(2\cos(q_j/2)t) & \text{for even } \alpha \\ \prod_{j=1}^{N} \sin(2\cos(q_j/2)t) & \text{for odd } \alpha \end{cases}.$$

All quantities are periodic in α with period 2, so we can set $A = 2$ in (3.20) leading to

$$\mathcal{L}(t) = \left[\prod_{j=1}^{N} \cos(2\cos(q_j/2)t) + \prod_{j=1}^{N} \sin(2\cos(q_j/2)t) \right]. \tag{A.6}$$

A.2 Initial state $|\psi_2\rangle_B = |2, 2, 2, 2, \dots\rangle_B$

In this case we have $N = 2L$, only the $P = 0$ sector contributes and

$$x_j = \begin{cases} (j+1)/2 & \text{for odd } j \\ j/2 & \text{for even } j \end{cases}.$$

We have the definitions

$$q_j = \frac{2\pi}{2L} j - \frac{\pi}{2L}, \quad j = 1 \dots 2L \quad \text{and} \quad p_K = \frac{2\pi}{3L} K - \frac{\pi}{3L}, \quad K = 1 \dots 3L. \tag{A.7}$$

The matrix H is $2L \times 3L$ and its elements are

$$H_{jK} = \frac{1}{\sqrt{N}}(1 - e^{i(q_j - p_K)}) \sum_{m=1}^{L} e^{-iq_j 2m} e^{ip_K 3m} = \frac{\sqrt{N}}{2}(1 - e^{i(q_j - p_K)})(\delta_{j,K} \bmod L). \tag{A.8}$$

For the product in (3.20) we obtain a block-diagonal matrix, whose determinant is

$$\det(H\Lambda H^\dagger) = \left(\frac{L}{2}\right)^N \prod_{j=1}^{L} \det M^j, \tag{A.9}$$

where each M^j is a 2×2 matrix with elements given by

$$M^j_{ab} = \sum_{c=0}^{2} (1 - e^{i(q_{j+aL} - p_{j+cL})})(1 - e^{-i(q_{j+bL} - p_{j+cL})}) \Lambda_{j+cL, j+cL} \tag{A.10}$$

For the sub-determinants we get

$$\begin{aligned} \det M^j &= \sum_{c,d=0}^{2} 16 \sin\left(\frac{q_j - p_{j+cL}}{2}\right) \cos\left(\frac{q_j - p_{j+dL}}{2}\right) \sin\left(\frac{p_{j+dL} - p_{j+cL}}{2}\right) \Lambda_{j+cL} \Lambda_{j+dL} \\ &= \sum_{c,d=0}^{2} 16 \sin\left(\frac{q_j - p_{j+cL}}{2}\right) \cos\left(\frac{q_j - p_{j+dL}}{2}\right) \sin\left(\frac{(c-d)\pi}{3}\right) \Lambda_{j+cL} \Lambda_{j+dL} \\ &= \sum_{c,d=0}^{2} 16 \sin\left(\frac{q_j}{6} - \frac{c\pi}{3}\right) \cos\left(\frac{q_j}{6} - \frac{d\pi}{3}\right) \sin\left(\frac{(c-d)\pi}{3}\right) e^{2i\alpha p_j} e^{i\alpha(c+d)2\pi/3} e^{-i(E(p_{j+cL}) + E(p_{j+dL}))t}. \end{aligned}$$

All quantities are α-periodic with a period of 3, therefore we can set $A = 3$:

$$\mathcal{L}(t) = \sum_{\alpha=1}^{3} e^{i\alpha L 2\pi/3} \prod_{j=1}^{L} z_\alpha(q_j),$$

where

$$z_\alpha(q_j) = \frac{4}{9}\sum_{c,d=0}^{2}\left[\sin\left(\frac{q_j}{6}-\frac{c\pi}{3}\right)\cos\left(\frac{q_j}{6}-\frac{d\pi}{3}\right)\sin\left(\frac{(c-d)\pi}{3}\right)\times\right.$$
$$\left.\times\, e^{i\alpha\frac{(c+d)2\pi}{3}}e^{2it\left(\cos\left(\frac{2q_j+c2\pi}{3}\right)+\cos\left(\frac{2q_j+d2\pi}{3}\right)\right)}\right].$$

In the thermodynamic limit only the term with $\alpha = 3$ survives and this leads to

$$\lim_{L\to\infty}\frac{1}{L}\log|\mathcal{L}(t)| = \frac{1}{\pi}\int_0^\pi dq\ \log|z(q)|, \tag{A.11}$$

where

$$z(q) = \frac{4}{9}\sum_{c,d=0}^{2}\left[\sin\left(\frac{q}{6}-\frac{c\pi}{3}\right)\cos\left(\frac{q}{6}-\frac{d\pi}{3}\right)\sin\left(\frac{(c-d)\pi}{3}\right)e^{2it\left(\cos\left(\frac{2q+c2\pi}{3}\right)+\cos\left(\frac{2q+d2\pi}{3}\right)\right)}\right].$$

B Direct equivalence of the bosonic and fermionic calculations of the EFP

Here we prove that the fermionic results for the EFP can be obtained directly from the bosonic formula (3.24). We require that the initial state is translationally invariant. In order to conform with the notations in the main text we will use the convention that the lower and upper case indices take values $1,\ldots,N$ and $1,\ldots,(N+L)$, respectively.

We introduce the $(N+L)\times(N+L)$ Fourier-transform matrix as

$$U_{IJ} = \frac{1}{\sqrt{N+L}}e^{ip_J I},$$

where the p_J variables are defined in (3.14). It is easy to see that U is unitary. Note that this matrix is different from V introduced in (3.18), which performs a Fourier transform over N points.

Our goal is to calculate the EFP from expression (3.32). In the present calculation it is useful to treat the two factors in the Λ matrices separately. Inserting $1 = UU^\dagger = U^\dagger U$ we write with some abuse of notation

$$O_m = \tilde{G}\Lambda_1 F_m \Lambda_2 \tilde{G}^\dagger = \tilde{G}e^{i\alpha_1 p}U^\dagger U e^{iE(p)t}F_m e^{-iE(p)t}U^\dagger U e^{-i\alpha_2 p}\tilde{G}^\dagger.$$

It is easy to see that

$$(\tilde{G}e^{i\alpha_{1,2}p}U^\dagger)_{kJ} = \sqrt{N+L}\,\delta_{J,k+x_k+\alpha_{1,2}}.$$

Using (3.29) we write

$$\left(e^{iE(p)t}F_m e^{-iE(p)t}\right)_{JK} = (N+L)\delta_{JK} - \sum_{\beta=1}^{m+1}\frac{e^{iE(p_J)t}}{e^{iE(p_K)t}}\frac{e^{i\beta p_J}}{e^{i\beta p_K}}.$$

Multiplying with U and $U^\dagger$ we get

$$\left(Ue^{iE(p)t}F_m e^{-iE(p)t}U^\dagger\right)_{JK} = (N+L)\delta_{JK} - (N+L)\sum_{\beta=1}^{m+1}U^*_{J+\beta}(t)U_{K+\beta}(t),$$

where

$$U_\beta(t) = \frac{1}{N+L}\sum_{J=1}^{N+L}e^{-i\beta p_J}e^{-iE(p_J)t},$$

which is equal to the fermionic propagator introduced in (4.13), up to an irrelevant constant shift in the energy eigenvalue.

Finally

$$O_{m,jk} = (N+L)^2 \left[\delta_{jk}\delta_{\alpha_1,\alpha_2} - \sum_{\beta=1}^{m+1} U^*_{j+x_j+\alpha_1+\beta}(t) U_{k+x_k+\alpha_2+\beta}(t) \right].$$

If $\alpha_1 \neq \alpha_2$ then the determinant vanishes except in the very small chain with $N = m+1$. We don't consider this case and set $\alpha_1 = \alpha_2$.

The determinant of O_m can be computed using the following general identity, which is valid for arbitrary $\kappa \leq N$:

$$\det_N(\delta_{jk} + \sum_{\beta=1}^{\kappa} a_j^{(\beta)} b_k^{(\beta)})_{j,k=1..N} = \det_\kappa(\delta_{\beta,\gamma} + \sum_{j=1}^{N} a_j^{(\beta)} b_j^{(\gamma)})_{\beta,\gamma=1..\kappa}. \tag{B.1}$$

In the present case we get

$$\det O_m = (N+L)^{2N} \det_{m+1} K,$$

where

$$K_{\beta\gamma} = \delta_{\beta\gamma} - \sum_{j=1}^{N} U^*_{j+x_j+\alpha+\beta}(t) U_{j+x_j+\alpha+\gamma}(t).$$

If the coordinates $y_j = j + x_j$ form a set of periodicity ν, then we can set $A = \nu$ and obtain

$$\langle E_m(t) \rangle = \frac{(1+n)^2}{\nu^2} \sum_{\alpha=1}^{\nu} \det_{m+1} K. \tag{B.2}$$

If the initial state is $|\psi_n\rangle_B$ then $\nu = 1+n$ and the sum in (B.2) describes the averaging over the $1+n$ components of the fermionic state $|\psi_n\rangle_F = \tilde{P}|\psi_n\rangle_B$. It is easy to check that (B.2) exactly agrees with formula (4.17), which was derived using free fermion techniques.

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