# TRIMMED STABLE AR(1) PROCESSES

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ABSTRACT. In this paper we investigate the distribution of trimmed sums of dependent observations with heavy tails. We consider the case of autoregressive processes of order one with independent innovations in the domain of attraction of a stable law. We show if the $d$ largest (in magnitude) terms are removed from the sample, then the sum of the remaining elements satisfies a functional central limit theorem with random centering provided $d = d(n) \geq n^\gamma$ (for some $\gamma > 0$), $d(n)/n \to 0$. This result is used to get asymptotics for the widely used CUSUM process in case of dependent heavy tailed observations.

## 1. Introduction and results

Let $X_1, X_2, \ldots,$ be independent, identically distributed random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$. Lévy (1935) and Darling (1952) noted that the order of magnitude of the sum $S_n = \sum_{k=1}^n X_k$ is the same as that of its largest term and the contribution of a fixed, but large number of extremal terms is essentially responsible for the distribution of $S_n$. The asymptotic distribution of the trimmed sum $S_n^{(d)}$ obtained from $S_n$ by discarding the $d$ smallest and $d$ largest summands was determined by LePage et al. (1981) and Csörgő et al. (1986) proved that for $d(n) \to \infty$, $d(n)/n \to 0$ the trimmed sum $S_n^{(d)}$ satisfies the central limit theorem. Arov and Bobrov (1960), Mori (1984), Hall (1978), Teugels (1981), Griffin and Pruitt (1987, 1989) and Kesten (1993) considered a different type of trimming of the sample. Let $\eta_{n,d}$ denote the $d$–th largest element of $|X_1|, \ldots, |X_n|$. These authors were interested in the asymptotic behavior of the modulus trimmed sum $^{(d)}S_n = \sum_{k=1}^n X_k I\{|X_k| \leq \eta_{n,d}\}$, i.e. when from the sum we remove the $d$ elements with the largest absolute values. Griffin and Pruitt (1987) proved that the trimmed central limit theorem of Csörgő et al. (1986) remains valid for modulus trimmed sums provided the distribution of $X_1$ is symmetric, but it generally fails for nonsymmetric variables and it can happen that $^{(d)}S_n$ is asymptotically normal for some $d(n)$, but not for another $d'(n) \geq d(n)$. This is somewhat unexpected, since removing more large elements from the sample should result in better behavior. Sufficient conditions for the asymptotic normality of $^{(d)}S_n$ in the nonsymmetric case were given by Berkes and Horváth (2012). On the other hand, Berkes et

al. (2011a) showed that if $d(n) \to \infty$, $d(n)/n \to 0$, a functional central limit theorem always holds for ${}^{(d)}S_n$ with a random centering factor.

Trimming also has important applications in statistics. As an example, we consider the detection of possible changes in the location model

$$(1.1) \qquad\qquad X_j = c_j + e_j, \quad 1 \le j \le n,$$

where $e_1, \ldots, e_n$ are random errors. Under the null hypothesis of stability, the location parameter is constant, i.e.

$$H_0: \quad c_1 = c_2 = \ldots = c_n.$$

If $H_0$ holds, then

$$(1.2) \qquad\qquad X_j = c + e_j, \quad 1 \le j \le n,$$

with some constant $c$. Under the alternative there are $r$ changes:

$$H_A: \text{ there is } r \ge 1 \text{ and } 1 < k_1 < k_2 < \ldots < k_r < n \text{ such that}$$
$$c_1 = \ldots = c_{k_1-1} \ne c_{k_1} = c_{k_1+1} = \ldots = c_{k_2-1} \ne c_{k_2} = c_{k_2+1} = \ldots$$
$$= c_{k_r-1} \ne c_{k_r} = \ldots = c_n.$$

The most popular methods to test $H_0$ against $H_A$ (cf. Csörgő and Horváth (1998) and Aue and Horváth (2012)) are based on the CUSUM process

$$(1.3) \qquad\qquad U_n(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Clearly, if $H_0$ is true, then $U_n(t)$ does not depend on the common but unknown location parameter $c_1$. It is well known if $X_1, \ldots, X_n$ are independent and identically distributed random variables with a finite second moment, then

$$\frac{1}{(n \mathrm{var}(X_1))^{1/2}} U_n(x) \overset{\mathcal{D}[0,1]}{\longrightarrow} B(x),$$

where $B(x)$ is a Brownian bridge and $\overset{\mathcal{D}[0,1]}{\longrightarrow}$ means weak convergence in the space $\mathcal{D}[0,1]$ of cadlag functions equipped with the Skorokhod $J_1$ topology (cf. Billingsley (1968)). Assuming that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables in the domain of attraction of a stable law of index $\alpha \in (0,2)$, Aue et al. (2008) showed that

$$\frac{1}{n^{1/\alpha} \hat{L}(n)} U_n(x) \overset{\mathcal{D}[0,1]}{\longrightarrow} B_\alpha(x),$$

where $\hat{L}$ is a slowly varying function at $\infty$ and $B_\alpha(x)$ is an $\alpha$–stable bridge. (The $\alpha$–stable bridge is defined as $B_\alpha(x) = W_\alpha(x) - x W_\alpha(1)$, where $W_\alpha$ is a Lévy $\alpha$–stable motion.) Since

nothing is known on the distributions of the functionals of $\alpha$–stable bridges, Berkes et al. (2011a) suggested the trimmed CUSUM process

$$(1.4) \qquad T_{n,d}(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i I\{|X_i| \leq \eta_{n,d}\} - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i I\{|X_i| \leq \eta_{n,d}\}.$$

Assuming that the $X_i$'s are independent and identically distributed and are in the domain of attraction of a stable law, they proved

$$(1.5) \qquad \frac{1}{\sigma_n} T_{n,d}(x) \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where

$$\sigma_n^2 = \frac{\alpha}{2-\alpha}(H^{-1}(d/n))^2 d,$$

$B(t)$ is a Brownian bridge and $H^{-1}$ denotes the generalized inverse of $H$, the survival function of $X_1$. The CUSUM process has also been widely used in case of dependent variables but it is nearly always assumed that the observations have high moments and the dependence in the sequence is weak. For a review we refer to Aue and Horváth (2012). However, very few papers consider the instability of time series models with heavy tails.

Fama (1965) and Mandelbrot (1963, 1967) pointed out that the distributions of commodity and stock returns are often heavy tailed with possible infinite variance and their research started the investigation of time series models where the marginal distributions have regularly varying tails. Davis and Resnick (1985, 1986) investigated the properties of moving averages with regularly varying tails and obtained non–Gaussian limits for the sample covariances and correlations. Their results were extended to heavy tailed ARCH by Davis and Mikosch (1998). The empirical periodogram was studied by Mikosch et al. (2000). Andrews et al. (2009) estimated the parameters of autoregressive processes with stable innovations.

In this paper we study trimmed sums of AR(1) sequences with heavy tails. Let $e_i$ be a $\sigma(\varepsilon_j, j \leq i)$ measurable solution of

$$(1.6) \qquad e_i = \rho e_{i-1} + \varepsilon_i \quad -\infty < i < \infty.$$

We assume throughout this paper that

$$(1.7) \qquad \varepsilon_j, -\infty < j < \infty \text{ are independent and identically distributed,}$$

$$(1.8) \qquad \varepsilon_0 \text{ belongs to the domain of attraction of a stable}$$
$$\text{random variable } \xi^{(\alpha)} \text{ with parameter } 0 < \alpha < 2,$$

and

$$(1.9) \qquad \varepsilon_0 \text{ is symmetric when } \alpha = 1.$$

Assumption (1.8) means that

$$(1.10) \qquad \left(\sum_{j=1}^{n} \varepsilon_j - a_n\right)\bigg/ b_n \xrightarrow{\mathcal{D}} \xi^{(\alpha)}$$

for some numerical sequences $a_n$ and $b_n$. The necessary and sufficient condition for this is

$$(1.11) \qquad \lim_{t \to \infty} \frac{P\{\varepsilon_0 > t\}}{L_*(t)t^{-\alpha}} = p \quad \text{and} \quad \lim_{t \to \infty} \frac{P\{\varepsilon_0 \leq -t\}}{L_*(t)t^{-\alpha}} = q$$

for some numbers $p \geq 0$, $q \geq 0$, $p + q = 1$, where $L_*$ is a slowly varying function at $\infty$. It is known that (1.6) has a unique stationary non–anticipative solution if and only if

$$(1.12) \qquad -1 < \rho < 1.$$

Under assumptions (1.7)–(1.12), $\{e_j\}$ is a stationary sequence and $E|e_0|^\kappa < \infty$ for all $0 < \kappa < \alpha$ but $E|e_0|^\kappa = \infty$ for all $\kappa > \alpha$. AR(1) processes with stable innovations were considered by Chan and Tran (1989), Chan (1990), Aue and Horváth (2007) and Zhang and Chan (2010) who investigated the case when $\rho$ is close to 1 and provided estimates for $\rho$ and and other the parameters when the observations do not have finite variances.

The convergence of the finite dimensional distributions of $U_n(x)$ in the AR(1) case is an immediate consequence of Phillips and Solo (1992) representation. Let $\xrightarrow{fdd}$ denote the convergence of the finite dimensional distributions. If (1.2)–(1.9) and (1.12) hold, then we have

$$(1.13) \qquad \frac{1 - \rho}{n^{1/\alpha}L_*(n)}U_n(x) \xrightarrow{\text{fdd}} B_\alpha(x),$$

where $B_\alpha(x), 0 \leq x \leq 1$ is an $\alpha$–stable bridge and $L_*$ is defined in (1.11). It has been pointed out by Avram and Taqqu (1986, 1992) that the fdd convergence in (1.13) cannot be replaced with weak convergence in $\mathcal{D}[0,1]$. However, Avram and Taqqu (1992) proved that $U_n(x)$ converges in the weak–$M_1$ sense under some additional regularity conditions. Some of their regularity conditions were removed by Tyran–Kamińska (2010). For further results on the weak convergence of dependent sequence with infinite variance in the $M_1$ topology we refer to Basrak et al. (2012).

We formulate now our main results. On the truncation parameter $d = d(n)$ we will assume

$$(1.14) \qquad \lim_{n \to \infty} d(n)/n = 0$$

and

$$(1.15) \qquad d(n) \geq n^\delta \quad \text{with some} \quad 0 < \delta < 1.$$

Let $F(x) = P\{X_0 \leq x\}$, $H(x) = P\{|X_0| > x\}$ and let $H^{-1}(t)$ be the (generalized) inverse of $H$. Our last condition will be used to establish the weak law of large numbers for $\eta_{n,d}$. We assume that $\varepsilon_0$ has a density function $p(t)$ which satisfies

$$(1.16) \qquad \int_{-\infty}^{\infty} |p(t + s) - p(t)|dt \leq C|s| \quad \text{with some} \quad C.$$

Let

$$(1.17) \qquad A_n = d^{1/2}H^{-1}(d/n)$$

and

(1.18) $$m(t) = EX_1 I\{|X_1| \leq t\}.$$

**Theorem 1.1.** *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{1}{A_n} \sum_{k=1}^{n} [X_k I\{|X_k| \leq \eta_{n,d}\} - m(\eta_{n,d})] \overset{\mathcal{D}[0,1]}{\longrightarrow} W(x),$$

*where $W(x)$ is a Wiener process.*

The result in Theorem 1.1 uses the the random centering factor $m(\eta_{n,d})$. This factor is characteristic for the asymptotic distribution of the modulus trimmed partial sums process, as first observed in Berkes et al. (2011a). Since a random translation of the terms in the CUSUM process cancels out, the next result is an immediate consequence of Theorem 1.1.

**Theorem 1.2.** *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{T_{n,d}(x)}{A_n} \overset{\mathcal{D}[0,1]}{\longrightarrow} B(x),$$

*where $B(x)$ is a Brownian bridge.*

Statistical applications of Theorem 1.2 require the estimation of the norming factor from the observations. The estimation of this term will be studied in a subsequent paper.

## 2. **Preliminary results**

The proofs of Theorems 1.1 and 1.2 are based on several technical lemmas.

We can and will assume without loss of generality that

(2.1) $$E\varepsilon_0 = 0, \quad \text{if } 1 < \alpha < 2.$$

Under these conditions, in (1.10) we can choose $a_n = 0$ and $b_n$ can be chosen any sequence satisfying

(2.2) $$\frac{n}{b_n^\alpha} L_*(b_n) \to 1.$$

According to the result of Cline (1983) (cf. also Davis and Resnick (1986)), $H(x)$, the survival function of $|X_0|$ satisfies

(2.3) $$H(x) = x^{-\alpha} L(x),$$

where $L(x)$ is a slowly varying function at $\infty$ and

(2.4) $$\lim_{x \to \infty} \frac{H(x)}{P\{|\varepsilon_0| > x\}} = \lim_{x \to \infty} \frac{L(x)}{L_*(x)} = \frac{1}{1 - |\rho|^\alpha}.$$

Let

$$u_{k,n}(t) = X_k I\{|X_k| \leq tH^{-1}(d/n)\} \quad \text{and} \quad m_n(t) = E[X_0 I\{|X_0| \leq tH^{-1}(d/n)\}].$$

The main goal of this section is to get bounds for $Eu_0(t)u_k(s)$ and $\mathrm{cov}(u_0(t), u_k(s))$.

**Lemma 2.1.** *We assume that (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold. Let $\mathbf{Y}^{(k)} = (X_0, X_k)$ and let $\mathbf{Y}_i^{(k)}, i = 1, 2, \ldots$ be independent and identically distributed copies of $\mathbf{Y}^{(k)}$. Then*

$$\frac{\mathbf{Y}_1^{(k)} + \ldots + \mathbf{Y}_n^{(k)}}{n^{1/\alpha}L_*(n)} \xrightarrow{\mathcal{D}} \mathbf{Z}^{(k)} \quad as \quad n \to \infty,$$

*where $\mathbf{Z}^{(k)} = (Z_1^{(k)}, Z_2^{(k)})$ with*

$$Z_1^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{-\ell}^{(\alpha)} \quad and \quad Z_2^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{k-\ell}^{(\alpha)}$$

*and $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ are independent and identically distributed copies of $\xi^{(\alpha)}$.*

*Proof.* It follows from (1.6) that

$$(2.5) \qquad X_k - c = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell} = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} + \rho^k X_0, \quad 1 \leq k < \infty.$$

Let $\varepsilon_\ell^{(i)}, -\infty < \ell < \infty, i = 1, 2, \ldots$ be independent and identically distributed copies of $\varepsilon_0$. Clearly

$$\mathbf{Y}_i^{(k)} = (Y_{i,1}^{(k)}, Y_{i,2}^{(k)}) \quad \text{with} \quad Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad Y_{i,2}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell}^{(i)}$$

are independent and identically distributed copies of $\mathbf{Y}^{(k)}$. Elementary algebra gives

$$\sum_{i=1}^{n} Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^{n} \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad \sum_{i=1}^{n} Y_{i,2}^{(k)} = \sum_{\ell=0}^{k-1} \rho^\ell \sum_{i=1}^{n} \varepsilon_{k-\ell}^{(i)} + \rho^k \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^{n} \varepsilon_{-\ell}^{(i)}.$$

For every $L \geq 0$ by (1.10) we have that (recall that under our conditions the centering factors $a_n$ in (1.10) can be chosen 0)

$$\frac{1}{b_n} \left( \sum_{i=1}^{n} \varepsilon_\ell^{(i)}, -L \leq \ell \leq L \right) \xrightarrow{\mathcal{D}} \left( \xi_\ell^{(\alpha)}, -L \leq \ell \leq L \right),$$

where $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ are independent and identically distributed copies of $\xi^{(\alpha)}$. Let $0 < \kappa < \alpha$. It follows from de Acosta and Giné (1979) that

$$E \left| \frac{1}{b_n} \sum_{i=1}^{n} \varepsilon_\ell^{(i)} \right|^\kappa \leq C_1,$$

and therefore for every $x > 0$ we have that

$$\lim_{L\to\infty} \limsup_{n\to\infty} P\left\{\sum_{\ell=L+1}^{\infty} \rho^\ell \left|\frac{1}{b_n}\sum_{i=1}^{n}\varepsilon_\ell^{(i)}\right| > x\right\} = 0$$

and similarly

$$\lim_{L\to\infty} P\left\{\sum_{\ell=L+1}^{\infty} \rho^\ell |\xi_\ell^{(\alpha)}| > x\right\} = 0.$$

This completes the proof of the lemma. $\qquad\square$

Let $\mathbf{i}$ denote the imaginary unit.

**Lemma 2.2.** *Let $\mathbf{Y}$ be a stable vector variable with characteristic function $\psi(s,t)$. Then there exists a measure $\nu$ on the Borel sets of $R^2$ such that for some $\mathcal{C}_1, \mathcal{C}_2$ and any $\gamma > 0$*

$$\psi(s,t) = \exp\left\{\mathbf{i}(\mathcal{C}_1 s + \mathcal{C}_2 t) + \int_{|\mathbf{u}|>\gamma}(e^{\mathbf{i}(su_1+tu_2)} - 1)\nu(du_1, du_2)\right.$$
$$\left. + \int_{0<|\mathbf{u}|\leq\gamma}(e^{\mathbf{i}(su_1+tu_2)} - 1 - \mathbf{i}(su_1 + tu_2))\nu(du_1, du_2)\right\},$$

*where $\mathbf{u} = (u_1, u_2)$.*

The result can be found, for example, in Gikhman and Skorohod (1969, Chapter 5). $\nu$ is called the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Y}$. The stable vectors in our paper will be centered, i.e. $c_1 = c_2 = 0$.

**Lemma 2.3.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then we have*

$$\lim_{T\to\infty} \frac{T^{\alpha-2}}{L_*(T)} EX_0 I\{|X_0| \leq vT\} X_k I\{|X_k| \leq wT\} = \frac{\alpha}{2-\alpha}\frac{\rho^k}{1-|\rho|^\alpha}(\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

*Proof.* It follows from Theorem 4 of Resnick and Greenwood (1979) that

$$(2.6) \qquad\qquad \lim_{n\to\infty} nP\left\{\frac{(X_0, X_k)}{b_n} \in A\right\} = \nu(A),$$

where $b_n$ is defined in (2.2) and $A$ is any Borel set of $R^2$, not containing $(0,0)$, $\nu(A) < \infty$ and the $\nu$–measure of the boundary of $A$ is 0. Since $nL_*(b_n)/b_n^\alpha \to 1$, with the choice of $n = \lfloor T^\alpha/L_*(T)\rfloor$ we get from (2.6) that

$$(2.7) \qquad\qquad \lim_{T\to\infty} \frac{T^\alpha}{L_*(T)} P\{(X_0, X_k)/T \in \mathbf{A}\} = \nu(\mathbf{A}),$$

where $\nu$ is the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Z}^{(k)}$. By elementary arguments we conclude from (2.7)

$$\lim_{T\to\infty} \frac{T^{\alpha-2}}{L_*(T)} EX_0 I\{|X_0| \leq vT\} X_k I\{|X_k| \leq wT\} = \int_{-v}^{v}\int_{-w}^{w} xy\nu(dx, dy).$$

Since $\xi^{(\alpha)}$ is a stable random variable, its characteristic function can be written as $\exp(-\psi(t))$ and with this notation we get

$$E \exp(\mathbf{i}(sZ_1^{(k)} + tZ_2^{(k)})) = \exp\left(-\sum_{\ell=0}^{\infty} \psi(s\rho^\ell + t\rho^{k+\ell}) - \sum_{\ell=0}^{k-1} \psi(t\rho^\ell)\right).$$

If $\hat{\nu}_\ell$ denotes the Lévy measure associated with the characteristic function $\exp(-\psi(s\rho^\ell + t\rho^{k+\ell}))$ and $\tilde{\nu}_\ell$ corresponds to $\exp(-\psi(t\rho^\ell))$, then we have

$$\nu(\mathbf{A}) = \sum_{\ell=0}^{\infty} \hat{\nu}_\ell(\mathbf{A}) + \sum_{\ell=0}^{k-1} \tilde{\nu}_\ell(\mathbf{A}).$$

Hence

$$\int_{-v}^{v} \int_{-w}^{w} xy\nu(dx, dy) = \sum_{\ell=0}^{\infty} \int_{-v}^{v} \int_{-w}^{w} xy\hat{\nu}_\ell(dx, dy).$$

Next we note that there is a positive constant $A^*$ such that

$$\lim_{x\to\infty} \frac{P\{|\xi^{(\alpha)}| > x\}}{x^{-\alpha}} = A^*$$

and therefore by Bingham et al (1989, p. 346) we obtain that

$$\lim_{x\to\infty} \frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \le x\}}{x^2 P\{|\xi^{(\alpha)}| > x\}} = \frac{\alpha}{2-\alpha}$$

resulting in

$$\lim_{x\to\infty} \frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \le x\}}{x^{2-\alpha}} = A^* \frac{\alpha}{2-\alpha}.$$

The last relation implies

$$\lim_{T\to\infty} T^{\alpha-2} E\left[\rho^{2\ell+k}(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \le T\min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)})\}\right]$$
$$= A^* \frac{\alpha}{2-\alpha} \rho^{2\ell+k}(\min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)}))^{2-\alpha}.$$

We note that $\exp(-\psi(s\rho^\ell + t\rho^{k+\ell}))$ is the characteristic function of the vector $(\rho^\ell \xi^{(\alpha)}, \rho^{k+\ell}\xi^{(\alpha)})$, so repeating the arguments leading to (2.6) and (2.7) for this vector instead of $(X_0, X_k)$ we get

$$\lim_{T\to\infty} \rho^{k+2\ell} \frac{T^{\alpha-2}}{A^*} E\xi^{(\alpha)} I\{|\rho^\ell \xi^{(\alpha)}| \le vT\}\xi^{(\alpha)} I\{|\rho^{\ell+k}\xi^{(\alpha)}| \le wT\} = \int_{-v}^{v} \int_{-w}^{w} xy\hat{\nu}_\ell(dx, dy),$$

and therefore

$$\int_{-v}^{v} \int_{-w}^{w} xy\hat{\nu}_\ell(dx, dy) = \frac{\alpha}{2-\alpha} \rho^k |\rho|^{\alpha\ell}(\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

Summing for $\ell = 0, 1, \ldots$, we get Lemma 2.3.                                        □

**Lemma 2.4.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then for every $k = 0, 1, 2, \ldots$*

$$(2.8) \qquad \lim_{n \to \infty} \frac{nE(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))}{A_n^2} = \frac{\alpha}{2 - \alpha} \rho^k (\min(s, t|\rho|^{-k}))^{2-\alpha}.$$

*Proof.* If $1 < \alpha < 2$, then

$$(2.9) \qquad \lim_{n \to \infty} m_n(t) = EX_0 \quad \text{for any } t > 0.$$

If $0 < \alpha < 1$, then

$$(2.10) \qquad |m_n(t)| \leq \int_{-tH^{-1}(d/n)}^{tH^{-1}(d/n)} |x| dF(x)$$

$$= -\int_0^{tH^{-1}(d/n)} x dH(x) = -xH(x) \Big|_0^{tH^{-1}(d/n)} + \int_0^{tH^{-1}(d/n)} H(x) dx.$$

By (2.3) and Bingham et al. (1989, p. 26) we have for $0 < \alpha < 1$

$$(2.11) \qquad \lim_{y \to \infty} \frac{\displaystyle\int_0^y H(x) dx}{yH(y)/(1 - \alpha)} = 1,$$

and therefore

$$(2.12) \qquad m_n(t) = O\left( H^{-1}(d/n) \frac{d}{n} \right).$$

If $\alpha = 1$, by assumption $e_0$ is symmetric, so under (1.2) we have that $X_1 = e_1 + c_1$ and therefore

$$(2.13) \qquad m_n(t) = O(1) + E[e_0 I\{|X_0| \leq tH^{-1}(d/n)\}]$$

$$= O(1) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} x dP\{e_1 \leq x\}$$

$$= O\left( H^{-1}(d/n) \frac{d}{n} \right) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} P\{e_1 \leq x\} dx$$

$$= O\left( H^{-1}(d/n) \frac{d}{n} \log H^{-1}(d/n) \right).$$

Thus we get from (2.9)–(2.13) for all $0 < \alpha < 2$ that

$$(2.14) \qquad \frac{nm_n(s)m_n(t)}{A_n^2} \to 0.$$

Lemma 2.3 yields

$$\lim_{n \to \infty} \frac{n}{A_n^2} \frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} EX_0 I\{|X_0| \leq sH^{-1}(d/n)\} X_k I\{|X_k| \leq tH^{-1}(d/n)\}$$

$$= \frac{\alpha}{2-\alpha} \frac{\rho^k}{1-|\rho|^\alpha} (\min(s,t|\rho|^{-k}))^{2-\alpha}.$$

By (2.4) we have

$$\lim_{n\to\infty} \frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} = \frac{1}{1-|\rho|^\alpha},$$

which completes the proof of the lemma.                                    □

**Lemma 2.5.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, we have for all $1/2 \le s \le t \le 3/2$ and $0 \le x \le 1$ that*

$$\lim_{n\to\infty} \frac{1}{A_n^2} E \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(s) - m_n(s)) \right) \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(t) - m_n(t)) \right)$$

$$= x \frac{\alpha}{2-\alpha} \left( s^{2-\alpha} + \sum_{k=1}^\infty \rho^k [(\min(s,t|\rho|^{-k})^{2-\alpha} + \min(t,s|\rho|^{-k})^{2-\alpha}] \right).$$

*Proof.* We note that

$$E \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(s) - m_n(s)) \right) \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(t) - m_n(t)) \right)$$

$$= \lfloor nx \rfloor E(u_{0,n}(s) - m_n(s))(u_{0,n}(t) - m_n(t))$$

$$+ \sum_{k=1}^{\lfloor nx \rfloor - 1} (\lfloor nx \rfloor - k) E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))$$

$$+ \sum_{k=1}^{\lfloor nx \rfloor - 1} (\lfloor nx \rfloor - k) E(u_{0,n}(t) - m_n(t))(u_{k,n}(s) - m_n(s)).$$

Let

(2.15)                    $$e_k^* = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} \quad \text{and} \quad X_k^* = c_1 + e_k^*.$$

It follows from Cline(1983) that there is a constant $C_1$ such that

(2.16)                    $$P\{|X_k^*| > x\} \le C_1 x^{-\alpha} L(x) \quad \text{for all } k \text{ and } 0 \le x < \infty.$$

Clearly as in (2.5),

(2.17)          $$X_k - X_k^* = e_k - e_k^* = \sum_{\ell=k}^\infty \rho^\ell \varepsilon_{k-\ell} = \sum_{j=0}^\infty \rho^{k+j} \varepsilon_{-j} = \rho^k (X_0 - c_1).$$

Next we write

$$|E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))|$$

$$= |Eu_{0,n}(t)u_{0,n}(s) - m_n(t)m_n(s)|$$
$$\leq |E(X_{0,n}(X_k - X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\}|$$
$$+ |E(X_0 X_k^* I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\} - m_n(s)m_n(t)|$$
$$\leq A_{1,k,n} + A_{2,k,n} + A_{3,k,n}$$

with

$$A_{1,k,n} = E|X_0(X_k - X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\}|,$$
$$A_{2,k,n} = E\big[|X_0 X_k^*|I\{|X_0| \leq sH^{-1}(d/n)\}$$
$$\times \big|I\{|X_k| \leq tH^{-1}(d/n)\} - I\{|X_k^*| \leq tH^{-1}(d/n)\}\big|\big]$$

and

$$A_{3,k,n} = |E(X_0 X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k^*| \leq tH^{-1}(d/n)\} - m_n(s)m_n(t)|.$$

Using (2.14) and (2.17) we conclude

$$(2.18) \qquad A_{1,k,n} \leq |\rho|^k E|X_0||X_0 - c_1|I\{|X_0| \leq sH^{-1}(d/n)\}$$
$$\leq C_2|\rho|^k (H^{-1}(d/n))^2 d/n$$

with some constant $C_2$. Next we note that

$$(2.19) \qquad A_{2,k,n} \leq E\big[|X_0 X_k^*|I\{|X_0| \leq sH^{-1}(d/n)\}$$
$$\times I\{tH^{-1}(d/n) - |\rho|^k|X_0| \leq |X_k^*| \leq tH^{-1}(d/n)\}\big]$$
$$+ E\big[|X_0 X_k^*|I\{|X_0| \leq sH^{-1}(d/n)\}$$
$$\times I\{tH^{-1}(d/n) \leq |X_k^*| \leq tH^{-1}(d/n) + |\rho|^k|X_0|\}\big]$$
$$= A_{2,k,n}^{(1)} + A_{2,k,n}^{(2)}.$$

Using the independence of $X_0$ and $X_k^*$ we get

$$A_{2,k,n}^{(1)} \leq E|X_0|I\{|X_0| \leq sH^{-1}(d/n)\}$$
$$\times E|X_k^*|I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \leq |X_k^*| \leq tH^{-1}(d/n)\}.$$

By (2.16) we have that

$$(2.20) \quad E|X_k^*|I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \leq |X_k^*| \leq tH^{-1}(d/n)\}$$
$$= -xP\{|X_k^*| > x\}\Big|_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} + \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} P\{|X_k^*| > x\}dx$$
$$\leq \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)\}} P\{|X_k^*| > x\}dx$$
$$\leq C_3|\rho|^k H^{-1}(d/n)d/n,$$

where $C_3$ is a constant. Hence, on account of (2.9), (2.12) and (2.13) we obtain that with some constant $C_4$

$$A_{2,k,n}^{(1)} \leq C_4 \rho^k (H^{-1}(d/n))^2 d/n$$

and similarly

$$A_{2,k,n}^{(2)} \leq C_4 \rho^k (H^{-1}(d/n))^2 d/n,$$

resulting in

$$(2.21) \qquad A_{2,k,n} \leq C_5 \rho^k (H^{-1}(d/n))^2 d/n.$$

Using again the independence of $X_0$ and $X_k^*$ we get

$$A_{3,k,n} = |m_n(s)||EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} - m_n(t)|.$$

It is easy to see that

$$EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\}$$
$$= EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\}$$
$$+ EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}$$

and by the independence of $X_0$ and $X_k^*$ and (2.16) we have

$$|EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\}| \leq C_5 |m_n(t)| H(|\rho|^{-k/2} H^{-1}(d/n))$$
$$\leq C_6 |m_n(t)||\rho|^{k\alpha/2} d/n.$$

Next we note that

$$\left| E\left[ X_k^* I\{|X_k^*| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}\right] \right.$$
$$- E\left[ (X_k^* + \rho^k(X_0 - c_1)) I\{|X_k^* + \rho^k(X_0 - c_1))| \leq tH^{-1}(d/n), \right.$$
$$\left. \left. |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}\right] \right|$$
$$\leq |\rho|^k E\left[ |X_0 - c_1| I\{|X_k^* + \rho^k(X_0 - c_1))| \leq tH^{-1}(d/n), \right.$$
$$\left. |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}\right]$$
$$+ E\left[ |X_k^*| |I\{|X_k^*| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\} \right.$$
$$\left. - I\{|X_k^* + \rho^k(X_0 - c_1))| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}|\right]$$
$$\leq |\rho|^k (|\rho|^{-k/2} H^{-1}(d/n) + |c_1|)$$
$$+ E|X_k^*| I\{(t - |\rho|^{k/2}) H^{-1}(d/n) - |c_1||\rho|^k \leq |X_k^*| \leq tH^{-1}(d/n)\}$$
$$+ E|X_k^*| I\{tH^{-1}(d/n) \leq |X_k^*| \leq (t + |\rho|^{-k/2}) H^{-1}(d/n) + |c_1||\rho|^k\}$$
$$\leq C_7 (|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n)$$

by (2.20). Similarly

$$|EX_k I\{|X_k| \leq tH^{-1}(d/n)\} - EX_k I\{|X_k| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}|$$
$$\leq C_8 (|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n).$$

Hence

(2.22) $$A_{3,k,n} \leq C_9 |\rho|^{\tau k} (H^{-1}(d/n))^2 d/n, \quad \text{where} \quad \tau = \min\{1, \alpha\}/2.$$

Putting together (2.18), (2.21) and (2.22) we get that

(2.23) $$\lim_{K \to \infty} \limsup_{n \to \infty} \frac{1}{A_n^2} \sum_{k=K}^{\lfloor nx \rfloor - 1} |(\lfloor nx \rfloor - k) E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))| = 0.$$

The lemma now follows from Lemma 2.4 and (2.23). $\qquad\square$

## 3. A weak convergence result

Define the two–parameter process

$$L_n(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_i I\{|X_i| \leq t H^{-1}(d/n)\} - m_n(t)),$$

for $0 \leq x \leq 1, 1/2 \leq t \leq 3/2$. First we show the tightness of $L_n(t)$. The proof is based on a generalization of Lemma 6 in Berkes et al (2011b). We introduce

$$X_{i,1} = \max(X_i, 0), \qquad X_{i,2} = \min(X_i, 0)$$

and

$$m_{n,1}(t) = EX_{0,1} I\{|X_0| \leq t H^{-1}(d/n)\}, \qquad m_{n,2}(t) = EX_{0,2} I\{|X_0| \leq t H^{-1}(d/n)\}.$$

Similarly to $L_n(t, x)$, we define

$$L_{n,1}(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_{i,1} I\{|X_i| \leq t H^{-1}(d/n)\} - m_{n,1}(t)),$$

and $L_{n,2}(t, x)$ is defined in a similar fashion. Clearly, if both $L_{n,1}$ and $L_{n,2}$ are tight, then $L_n(t, x)$ is tight as well. We prove only tightness of $L_{n,1}$, the same argument can be used in case of $L_{n,2}$. Let

$$g_n = \frac{1}{d^{1/2} \log \log n}.$$

**Lemma 3.1.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then*

(3.1) $$m_{n,1}(t) \quad \text{is a non-decreasing function on} \quad [1/2, 3/2],$$

(3.2) $$\frac{n}{A_n} \sup_{|t_2 - t_1| \leq g_n} |m_{n,1}(t_2) - m_{n,1}(t_1)| \to 0, \quad n \to \infty,$$

(3.3) $$E|L_{n,1}(t_2, x) - L_{n,1}(t_1, x)|^6 \leq C_1 |t_2 - t_1|^\tau, \quad \text{if} \quad |t_2 - t_1| \geq g_n,$$

*and*

(3.4) $$E|L_{n,1}(t, x_2) - L_{n,1}(t, x_1)|^6 \leq C_1 |x_2 - x_1|^\tau, \quad \text{if} \quad |x_2 - x_1| \geq g_n,$$

*with some $\tau > 2$ and constant $C_1$.*

*Proof.* The definition of $m_{n,1}(t)$ implies immediately (3.1).

By the definition of $m_{n,1}(t)$ we have for all $1/2 \le t_1 \le t_2 \le 3/2$ that

$$0 \le m_{n,1}(t_2) - m_{n,1}(t_1) = EX_{0,1}(I\{t_1 H^{-1}(d/n) < |X_0| \le t_2 H^{-1}(d/n)\})$$

$$\le \int_{t_1 H^{-1}(d/n)}^{t_2 H^{-1}(d/n)} x\, dH(x)$$

$$\le C_2 \Bigg( |t_2 H^{-1}(d/n) H(t_2 H^{-1}(d/n)) - t_1 H^{-1}(d/n) H(t_1 H^{-1}(d/n))|$$

$$+ |t_2 - t_1| H^{-1}(d/n) H(t_1 H^{-1}(d/n)) \Bigg)$$

$$\le C_3 |t_2 - t_1| \frac{d}{n} H^{-1}(d/n)$$

on account of integration by parts and (2.3), establishing (3.2).

Next we introduce

$$(3.5) \qquad Y_i = \sum_{k=0}^{\lfloor K \log n \rfloor} \rho^k \varepsilon_{i-k} + c_1, \qquad Y_{i,1} = \max(Y_i, 0)$$

and $\xi_i = \eta_i - E\eta_i$ with

$$\eta_i = \eta_i(t_1, t_2) = Y_{i,1} I\{t_1 H^{-1}(d/n) < |Y_i| \le t_2 H^{-1}(d/n)\}.$$

Since $E|\varepsilon_0|^{\alpha/2} < \infty$, using Markov's inequality we see that for every $\beta > 0$ there is a constant $K = K(\beta)$ such that

$$(3.6) \qquad \left| E(L_{n,1}(t_2, x) - L_{n,1}(t_1, x))^6 - \frac{1}{A_n^6} \sum_{1 \le i_1, \ldots, i_6 \le \lfloor nx \rfloor} E\xi_{i_1} \ldots \xi_{i_6} \right| \le C_5 n^{-\beta}.$$

We note that by definition, $\{\xi_i\}$ is a stationary, $\lfloor K \log n \rfloor$–dependent sequence with zero mean. Let us divide the indices $i_1, \ldots, i_6$ into groups so that the difference between the indices within a group are less than $\lfloor K \log n \rfloor$ and between groups is larger than $\lfloor K \log n \rfloor$. Clearly $E\xi_{i_1} \ldots \xi_{i_6} = 0$, if there is at least one group containing a single element. So it suffices to consider the cases when all groups contain at least two elements. This allows the cases of one single group with 6 elements ($D_1$), two groups with 3+3 ($D_2$) or 4+2 ($D_3$) elements and finally 3 groups with 2 elements in each ($D_4$). If there is only one group, then via Hölder's inequality we have

$$|E\xi_{i_1} \ldots \xi_{i_6}| \le E|\xi_0|^6 \le 2^6 (E|\eta_0|^6 + |E\eta_0|^6)$$

Since the cardinality of $D_1$ is bounded by constant times $n(\log n)^5$ we conclude

$$\left| \frac{1}{A_n^6} \sum_{D_1} E\xi_{i_1} \dots \xi_{i_6} \right|$$

$$\leq C_6 \left( \frac{n(\log n)^5}{A_n^6} [EX_0^6 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\} \right.$$

$$\left. + (EX_0 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\})^6] + n^{-\beta} \right).$$

Integration by parts and (2.3) yield

$$EX_0^6 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\} \leq C_7 |t_2 - t_1| \frac{d}{n} (H^{-1}(d/n))^6,$$

resulting in

$$\left| \frac{1}{A_n^6} \sum_{D_1} E\xi_{i_1} \dots \xi_{i_6} \right| \leq C_8 \left( \frac{(\log n)^5}{d^2} |t_2 - t_1| + n^{-\beta} \right).$$

Using again the $\lfloor K \log n \rfloor$ dependence of $\{\xi_i\}$ and the fact that the cardinality of $D_2$ is constant times $n^2(\log n)^4$ we conclude via Hölder's inequality

$$\left| \frac{1}{A_n^6} \sum_{D_2} E\xi_{i_1} \dots \xi_{i_6} \right|$$

$$= \left| \frac{1}{A_n^6} \sum_{D_2} E\xi_{i_1} \xi_{i_2} \xi_{i_3} E\xi_{i_4} \xi_{i_5} \xi_{i_6} \right|$$

$$\leq C_8 \left( \frac{n^2(\log n)^4}{A_n^6} [EX_0^3 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\} \right.$$

$$\left. + (EX_0 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\})^3]^2 + n^{-\beta} \right)$$

$$\leq C_9 \left( \frac{(\log n)^4}{d} (t_2 - t_1)^2 + n^{-\beta} \right).$$

Similar arguments give

$$\left| \frac{1}{A_n^6} \sum_{D_3} E\xi_{i_1} \dots \xi_{i_6} \right| \leq C_{10} \left( \frac{(\log n)^4}{d} (t_2 - t_1)^2 + n^{-\beta} \right).$$

Following the proof of Lemma 2.5 we obtain

$$\left| \frac{1}{A_n^6} \sum_{D_4} E\xi_{i_1} \dots \xi_{i_6} \right| \leq C_{11} \left( \frac{1}{A_n^6} \left( n \sum_{i=0}^{\infty} \xi_0 \xi_i \right)^3 + n^{-\beta} \right)$$

$$\leq C_{11} \left( |t_2 - t_1|^3 + n^{-\beta} \right).$$

Putting together our estimates and using the choice of $g_n$ we conclude for all $|t_2 - t_1| \geq g_n$

$$E(L_{n,1}(t_2, x) - L_{n,1}(t_2, x))^6 \leq C_{12}\left(\frac{(\log n)^5}{d^2}|t_2 - t_1| + \frac{(\log n)^4}{d}|t_2 - t_1|^2 + |t_2 - t_1|^3 + n^{-\beta}\right)$$
$$\leq C_{13}|t_2 - t_1|^\tau$$

with any $2 < \tau \leq 3$ on account of assumption (1.15). Hence the proof of (3.3) is complete. The proof of (3.4) goes along the lines of the arguments used to establish (3.3) and therefore it is omitted. $\square$

**Lemma 3.2.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then $L_n(t, x)$ is tight in $\mathcal{D}([1/2, 3/2] \times [0, 1])$.*

*Proof.* It follows from a minor modification of Lemma 6 in Berkes et al (2011b) that both $L_{n,1}$ and $L_{n,2}$ are tight. Since $L_n = L_{n,1} + L_{n,2}$, the result is proven. $\square$

Next we consider the convergence of the finite dimensional distributions. It is based in the following lemma:

**Lemma 3.3.** *We assume that (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold. Let $N = \lfloor(\log n)^\gamma\rfloor$ with some $\gamma > 0$. Then*

$$(3.7) \qquad E\left(\sum_{i=1}^N (X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}]\right)^4$$
$$\leq C_{13}\left(N(\log N)^3(H^{-1}(d/n))^4\frac{d}{n} + N^2(H^{-1}(d/n))^4\left(\frac{d}{n}\right)^2\right)$$

*with some constant $C_{13}$ and*

$$(3.8) \qquad \lim_{n\to\infty}\frac{Nn}{A_n^2}E\left(\sum_{k=1}^N (u_{k,n}(s) - m_n(s))\right)\left(\sum_{k=1}^N (u_{k,n}(t) - m_n(t))\right)$$
$$= \frac{\alpha}{2-\alpha}\left(s^{2-\alpha} + \sum_{k=1}^\infty \rho^k[(\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}]\right).$$

*Proof.* We recall the definition of $\xi_i$ from the proof of Lemma 3.1. For any $\beta > 0$, choosing $K$ in the definition of $Y_i$ in (3.5) we get that

$$E\left(\sum_{i=1}^N (X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}]\right)^4 \leq C_{14}\left(E\left(\sum_{i=1}^N \xi_i\right)^4 + n^{-\beta}\right).$$

We write

$$E\left(\sum_{i=1}^N \xi_i\right)^4 = \sum_{i_1,\ldots,i_4}^N E\xi_{i_1}\ldots\xi_{i_4}.$$

We note again that the $\{\xi_i\}$ is a stationary $K \log n$ dependent sequence with $0$ mean. Let us divide the indices $i_1, \ldots, i_4$ into blocks so that the difference between the indices within a block is less than $K \log n$ and between blocks is larger than $K \log n$. Clearly $E\xi_{i_1} \ldots \xi_{i_4} = 0$, if there is at least one block containing only a single element. So we need to consider the cases of one single block with 4 elements ($D_1$) and two blocks with 2+2 elements ($D_2$). The number of the elements in $D_1$ is not greater than constant times $N(\log N)^3$ and as we showed in the proof of Lemma 3.1

$$E\xi_0^4 \leq C_{14}\left((H^{-1}(d/n))^4\frac{d}{n} + n^{-\beta}\right),$$

assuming that $K$ in (3.5) is sufficiently large. Hence

$$\left|\sum_{D_1}^N E\xi_{i_1}\ldots\xi_{i_4}\right| \leq C_{15}\left(N(\log N)^3(H^{-1}(d/n))^4\frac{d}{n} + n^{-\beta}\right).$$

As in the proof of Lemma 3.1 we get that

$$\left|\sum_{D_2}^N E\xi_{i_1}\ldots\xi_{i_4}\right| \leq C_{16}N^2\left(\sum_{i=0}|E\xi_0\xi_i|\right)^2$$

and

$$\sum_{i=0}^\infty |E\xi_0\xi_i| \leq \left(C_{17}(H^{-1}(d/n))^2\frac{d}{n} + n^{-\beta}\right),$$

completing the proof of (3.7). The proof of (3.8) goes along the lines of the arguments used to establish Lemma 2.5. $\square$

**Lemma 3.4.** *If (1.2)–(1.9), (1.12)–(1.15) and (2.1) hold, then*

$$L_n(t, x) \longrightarrow \Gamma(t, x) \quad \text{weakly in} \quad \mathcal{D}([1/2, 3/2]) \times [0, 1]),$$

*where $\Gamma(t, x)$ is a Gaussian process with $E\Gamma(t, x) = 0$ and*

$$E\Gamma(t, x)\Gamma(s, y)$$
$$= \min(x, y)\frac{\alpha}{2-\alpha}\left((\min(s, t))^{2-\alpha} + \sum_{k=1}^\infty \rho^k[(\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}]\right).$$

*Proof.* By Lemma 3.2, the process $L_n(t, x)$ is tight, so we need only to show the convergence of the finite dimensional distributions. By the Cramér–Wold device it is sufficient to prove the asymptotic normality of

$$Q_n = \sum_{j=1}^J \sum_{\ell=0}^L \mu_{j,\ell}(L_n(t_j, x_{\ell+1}) - L_n(t_j, x_\ell))$$

for all $J, L$, real coefficients $\mu_{j,\ell}, 1/2 \leq t_j \leq 3/2, 1 \leq j \leq J$, and $0 = x_0 < x_1 < \ldots < x_L < x_{L+1} = 1$. We recall the definition of $X_k^*$ from the proof of Lemma 2.5 (cf. (2.17)) and define

$$\bar{L}_n(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} - EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\}).$$

Choosing $K$ large enough in the definition of $X_k^*$, we get from the arguments used in the proof of Lemmas 2.5, 3.1 and 3.3 that

$$E(L_n(t, x) - \bar{L}_n(t, x))^2 \to 0.$$

So we need to establish only the asymptotic normality of

$$\bar{Q}_n = \sum_{\ell=0}^{L} \sum_{j=1}^{J} \mu_{j,\ell}(\bar{L}_n(t_j, x_{\ell+1}) - \bar{L}_n(t_j, x_\ell)).$$

Let

$$z_{k,\ell} = \sum_{j=1}^{J} \mu_{j,\ell}(X_k^* I\{|X_k^*| \leq t_j H^{-1}(d/n)\} - E[X_k^* I\{|X_k^*| \leq t_j H^{-1}(d/n)\}]).$$

Since for all $\ell$

$$E\left(\frac{1}{A_n} \sum_{k=1}^{\lfloor K \log n \rfloor} z_{k,\ell}\right)^2 \to 0,$$

by stationarity and the $\lfloor K \log n \rfloor$–dependence of $z_{k,\ell}$ for any $\ell$ we get that the variables

$$\frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor+1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell}, \quad 1 \leq \ell \leq L \text{ are asymprotically independent.}$$

By stationarity we have

$$\frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor+1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell} \overset{\mathcal{D}}{=} \frac{1}{A_n} \sum_{k=1}^{\lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor} z_{k,\ell}.$$

Let us divide the integers of $[1, \lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor]$ into consecutive blocks $R_1, V_1, R_2, V_2, \ldots, R_s, V_s$ such that for $1 \leq i \leq s-1$, $R_i$ contains $\lfloor (\log n)^\gamma \rfloor$ integers, $V_i$ contains $\lfloor K \log n \rfloor$ integers, the last two blocks might contain less elements. Let

$$\zeta_{i,1} = \sum_{k \in R_i} z_{k,\ell} \quad \text{and} \quad \zeta_{i,2} = \sum_{k \in V_i} z_{k,\ell}.$$

Due to the $\lfloor K \log n \rfloor$ dependence and stationarity, the variables $\zeta_{i,2}, 1 \leq i < s$ are independent and identically distributed and the proof of Lemma 2.5 shows that

$$E\left(\frac{1}{A_n} \sum_{i=1}^{s} \zeta_{i,2}\right)^2 \to 0.$$

Using Lemma 3.3 we get that

$$E\zeta_{i,1}^2 \geq C_{18}(\log n)^\gamma (H^{-1}(d/n))^2 d/n$$

and

$$E\zeta_{i,1}^2 \leq C_{19}\left((\log n)^\gamma(\log\log n)^3(H^{-1}(d/n))^4\frac{d}{n} + (\log n)^{2\gamma}(H^{-1}(d/n))^4\left(\frac{d}{n}\right)^2\right).$$

Since $s$ is proportional to $n/(\log n)^\gamma$, a simple calculation yields

$$\frac{\sum_{i=1}^s E\zeta_{i,1}^4}{\left(\sum_{i=1}^s E\zeta_{i,1}^2\right)^2} \to 0,$$

Thus the central limit theorem with Lyapunov's remainder term (cf. Petrov 1995, p. 154) implies the asymptotic normality of $\sum_{1\leq k\leq\lfloor nx_{\ell+1}\rfloor-\lfloor nx_\ell\rfloor} z_{k,\ell}$. This completes the proof of Lemma 3.4. $\square$

## 4. Proof of Theorems 1.1 and 1.2

We need the weak law of large numbers for $\eta_{d,n}$.

**Lemma 4.1.** *If (1.2)–(1.9) and (1.12)–(1.16) hold, then we have*

$$\frac{\eta_{d,n}}{H^{-1}(d/n)} \xrightarrow{P} 1$$

*Proof.* Using Gorodetskii (1977) and Withers (1981) we get that $X_k$ is a strongly mixing stationary sequence with mixing rate $\alpha(k) \leq C_1\exp(-\lambda k)$ for some $C_1 > 0$ and $\lambda > 0$. Fix $1/2 < t < 2$ and let $T_k = I\{|X_k| \geq tH^{-1}(d/n)\}, 1 \leq k \leq n$. Clearly, $ET_k = P\{|X_k| \geq tH^{-1}(d/n)\} = H(tH^{-1}(d/n))$ and due to the the regular variation of $H$, $ET_k/(d/n) \to t^{-\alpha}$, as $n \to \infty$. On the other hand, by the correlation inequality of Davydov (1968) we get for any $p > 2$ that

$$|ET_0T_k - ET_0ET_k| \leq (\alpha(k))^{(p-1)/p}(ET_0^p)^{1/p}(ET_k^p)^{1/p}$$
$$\leq C_1\exp(-\lambda k(p-1)/p)(ET_0^p)^{2/p}$$
$$= C_1\exp(-\lambda k(p-1)/p)(ET_0)^{2/p}$$
$$\leq C_2\exp(-\lambda k(p-1)/p)(d/n)^{2/p}.$$

Hence setting $\bar{T}_k = T_k - ET_k$ we conclude that

$$E\left(\sum_{k=1}^n \bar{T}_k\right)^2 = nE\bar{T}_0^2 + 2\sum_{k=1}^{n-1}(n-k)E\bar{T}_0\bar{T}_k$$

$$\leq n \left( E\bar{T}_k^2 + 2 \sum_{k=1}^{n-1} |E\bar{T}_0\bar{T}_k| \right)$$

$$\leq n \left( ET_0^2 + C_3 \sum_{k=1}^{n-1} \exp(-\lambda k(p-1)/p)(d/n)^{2/p} \right)$$

$$\leq n \left( ET_0 + C_5(d/n)^{2/p} \right)$$

$$\leq n(d/n)^{2/p}.$$

Thus by Markov's inequality we have that

$$P\left\{ \sum_{k=1}^n \bar{T}_k \geq d^{2/p} \right\} \leq C_6 n^{(p-2)/p}/d^{2/p} \to 0,$$

provided that $d/n^{(2-p)/p} \to 0$. Since $d \geq n^\delta$, choosing $p$ near 2, it follows that

$$\sum_{k=1}^n T_k = t^{-\alpha}d(1 + o_P(1)) + o_P(d^{2/p}) = t^{-\alpha}d(1 + o_P(1)).$$

In other words,

$$\frac{1}{d}\#\{k \leq n : |X_k| \geq tH^{-1}(d/n)\} \xrightarrow{P} t^{-\alpha}, \quad \text{as} \quad n \to \infty.$$

This shows that

$$\lim_{n\to\infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 1 \quad \text{for} \quad t < 1$$

and

$$\lim_{n\to\infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 0 \quad \text{for} \quad t > 1,$$

completing the proof of Lemma 4.1.                                        □

*Proof of Theorem 1.1.* We note that $\Gamma(t, x)$ is a continuous process. Hence combining Lemmas 3.4 and 4.1 we conclude

$$L_n(\eta_{d,n}/H^{-1}(d/n), x) \xrightarrow{\mathcal{D}[0,1]} \Gamma(1, x).$$

It is easy to see that

$$\{\Gamma(1, x), 0 \leq x \leq 1\} \stackrel{\mathcal{D}}{=} \left\{ \left( \frac{\alpha}{2-\alpha} \frac{1+\rho}{1-\rho} \right)^{1/2} W(x), 0 \leq x \leq 1 \right\},$$

where $W(x)$ is a Wiener process, which completes the proof.                □

*Proof of Theorem 1.2.* Since

$$\frac{1}{A_n}T_{n,d}(x) = L_n(\eta_{d,n}/H^{-1}(d/n), x) - \frac{\lfloor nx \rfloor}{n}L_n(\eta_{d,n}/H^{-1}(d/n), 1),$$

Theorem 1.1 yields

$$\frac{1}{A_n} T_{n,d}(x) \quad \xrightarrow{\mathcal{D}[0,1]} \quad \left( \frac{\alpha}{2-\alpha} \frac{1+\rho}{1-\rho} \right)^{1/2} (W(x) - xW(1)).$$

By definition, $B(x) = W(x) - xW(1), 0 \le x \le 1$ is a Brownian bridge, so the proof of Theorem 1.2 is complete. □

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