On the asymptotic normality of kernel estimators of the long run covariance of functional time series

István Berkes

Institute of Statistics, Graz University of Technology, Graz, Austria, and The Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary.

Lajos Horváth

Department of Mathematics, University of Utah, Salt Lake City, Utah, USA

Gregory Rice

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada

Abstract

We consider the asymptotic normality in L^2 of kernel estimators of the long run covariance of stationary functional time series. Our results are established assuming a weakly dependent Bernoulli shift structure for the underlying observations, which contains most stationary functional time series models, under mild conditions. As a corollary, we obtain joint asymptotics for functional principal components computed from empirical long run covariance operators, showing that they have the favorable property of being asymptotically independent.

Keywords: functional time series, long run covariance operator, normal approximation, moment inequalities, empirical eigenvalues and eigenfunctions

1. Introduction

In multivariate time series analysis, the matrix valued spectral density and the long run covariance matrix, which is 2π times the spectral density evaluated at frequency zero, are fundamental in a multitude of applications. For example, the long run covariance matrix must be estimated in most inference problems related to the mean of stationary finite dimensional time series, see e.g. Hannan (1970), Xiao & Wu (2012), Politis (2011), and Aue et al. (2009). Additionally, dynamic principal component analysis utilizes estimates of the long run covariance matrix as well as the spectral density to perform meaningful dimension reduction for time series data, see Brillinger (2001).

Multivariate techniques are difficult to apply, however, when the data is obtained by observing a continuous time phenomena at a high resolution or at irregularly spaced time

---

points. A flexible alternative for studying such records is to break them at natural points, for example into daily or monthly segments, in order to form a series of curves. The field of functional time series analysis has grown considerably in recent years to provide methodology for such data; the main difference from traditional functional data analysis being that it accommodates for possible serial dependence. The long run covariance function, which is an analog of the long run covariance matrix, also plays a crucial role in this setting. We refer to Ferraty & Vieu (2006), Cuevas (2014), and Hsing & Eubank (2015) for a review of methods and the state of the art in functional data analysis, and Hörmann & Kokoszka (2012) for a survey on functional time series analysis.

In order to formally define the objects introduced above, let $\{X_i(t)\}_{i=-\infty}^{\infty}$, $t \in [0,1]$, be a stationary functional time series. The bivariate function

$$C(t,s) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell(t,s), \quad \text{where} \quad \gamma_\ell(t,s) = \text{cov}(X_0(t), X_\ell(s)),$$

is called the long run covariance function, and is a well defined element of $L^2([0,1]^2, \mathbb{R})$, assuming mild weak dependence conditions. $C(t,s)$ arises primarily as the asymptotic covariance of the sample mean function. Via right integration, $C(t,s)$ also defines a positive definite operator on $L^2([0,1], \mathbb{R})$ whose eigenvalues and eigenfunctions, or principal components, are the focus of a number of dimension reduction and inference techniques with dependent functional data. Due to its representation as a bi-infinite sum, $C(t,s)$ is naturally estimated with a kernel lag–window estimator of the form

$$\hat{C}_N(t,s) = \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \hat{\gamma}_i(t,s), \tag{1.1}$$

where

$$\hat{\gamma}_i(t,s) = \hat{\gamma}_{i,N}(t,s) = \begin{cases} \dfrac{1}{N} \sum_{j=1}^{N-i} (X_j(t) - \bar{X}_N(t))(X_{j+i}(s) - \bar{X}_N(t)), & i \geq 0 \\ \dfrac{1}{N} \sum_{j=1-i}^{N} (X_j(t) - \bar{X}_N(t))(X_{j+i}(s) - \bar{X}_N(t)), & i < 0. \end{cases}$$

We use the standard convention that $\hat{\gamma}_i(t,s) = 0$ when $i \geq N$.

The estimator in (1.1) was introduced in Horváth et al. (2013), where it is shown to be consistent under mild conditions, and its applications are developed in Horváth et al. (2014) and Jirak (2013) in the context of inference for the mean and stationarity testing with functional time series. Hörmann et al. (2013) develops an analog of dynamic principal component analysis based on the spectral density operator of functional time series, which is directly related to the long run covariance operator.

It is a classical result that kernel lag-window estimators of the spectral density of univariate and multivariate time series are, when suitably standardized, asymptotically normal; see Rosenblatt (1991). The definition of the spectral density operator of a stationary functional time series and its asymptotic normality were first established in the work

of Panaretos & Tavakoli (2013). In order to obtain their results, functional analogs of classical cummulant summability and mixing conditions are assumed. As noted in Shao & Wu (2007), cummulant conditions are exceedingly difficult to check, even with scalar time series, and mixing conditions, although classically popular, exhibit some unattractive pathologies. For example, the autoregressive one processes with independent and identically distributed errors that take the values 1 and -1 with equal probabilities are not mixing. In several theaters of application, non linear time series models are of interest, and in this case it is unknown whether such conditions are satisfied in the infinite dimensional setting.

In this paper, we establish the asymptotic normality of $\hat{C}_N(t,s)$ in $L^2([0,1]^2, \mathbb{R})$ for a broad class of stationary functional time series processes. In particular, we consider the case of $L^2([0,1], \mathbb{R})$ valued random functions exhibiting an $L^p - m$ approximable Bernoulli shift structure, which extends the results of Shao & Wu (2007) and Liu & Wu (2010) to the infinite dimensional setting. Doing so greatly generalizes the class of functional time series processes for which a normal approximation for $\hat{C}_N$ can be achieved. An immediate corollary of this result is the limit distribution of the empirical eigenvalues and eigenfunctions computed from $\hat{C}_N$, which play a fundamental role in principal component analysis with dependent data.

The rest of the paper is organized as follows. In Section 2, we state our assumptions and the main results of the paper. The section concludes with an application of our results to bandwidth selection. Section 3 contains the application to the limit distribution of the empirical eigenvalues and eigenfunctions computed from $\hat{C}_N$. The proofs of the main results of the paper are contained in Section 4, which is broken into several subsections that each illuminate the main techniques behind the proof.

2. Assumptions, and main results

2.1. Asymptotic normality of $\hat{C}_N$

Let $|| \cdot ||$ denote the L^2 norm of square integrable functions on $[0,1]^d$, the dimension $d \geq 1$ being clear by the input function, and let $\int$ to mean $\int_0^1$. Throughout this paper we assume that

$\mathbf{X} = \{X_i\}_{i=-\infty}^{\infty}$ forms a sequence of Bernoulli shifts, i.e. $X_j = g(\epsilon_j, \epsilon_{j-1}, ...)$ (2.1)
for some measurable function $g : S^\infty \mapsto L^2$ and iid random variables ϵ_j,
$-\infty < j < \infty$, with values in a measurable space S,

$$\epsilon_j(t) = \epsilon_j(t, \omega) \text{ is jointly measurable in } (t, \omega), \; -\infty < j < \infty \quad (2.2)$$

$$\mathrm{E}||X_0||^{4+\delta} < \infty \; \text{ for some } \; \delta > 0, \quad (2.3)$$

and

$$\{X_n\}_{n=-\infty}^{\infty} \text{ can be approximated by the } m\text{–dependent sequences} \quad (2.4)$$

3

$$X_{n,m} = g(\epsilon_n, \epsilon_{n-1}, ..., \epsilon_{n-m+1}, \boldsymbol{\epsilon}_{n,m}^*) \text{ with } \boldsymbol{\epsilon}_{n,m}^* = (\epsilon_{n,m,n-m}^*, \epsilon_{n,m,n-m-1}^*, \ldots),$$

where the $\epsilon_{n,m,k}^*$'s are independent copies of ϵ_0, independent of $\{\epsilon_i, -\infty < i < \infty\}$, such that

$$\sum_{k=1}^{\infty} w\left(\sum_{m=k}^{\infty} c_m\right) < \infty \quad \text{with} \quad c_m = (\mathrm{E}\|X_0 - X_{0,m}\|^4)^{1/4},$$

where $w(t) > 0$ is regularly varying at zero, and $w(t)/t^{1/3} \to 0$ as $t \to 0$.

We note that w is a regularly varying function at zero if $\lim_{t\to 0+} w(ut)/w(t) = u^\rho$ for some $\rho \geq 0$ and all $u > 0$. For a review of the properties of regularly varying functions we refer to Bingham (1989). Nearly all stationary time series models based on independent innovations satisfy condition (2.1), including linear processes in function spaces, and the functional ARCH and GARCH processes, see Bosq (2000), and Hörmann et al. (2015). Condition (2.4) specifies the level of dependence that is allowed within the sequence in terms of how well it can be approximated in the L^2 sense by finite dependent processes, and thus defines a version of L^p–m–approximability for functional time series, see Hörmann & Kokoszka (2010). Condition (2.4) is satisfied when, for example, $c_m = O(m^{-\alpha})$ for some $\alpha > 4$. In comparison, Shao & Wu (2007) assume a geometric rate of decay for similar approximation coefficients with scalar time series. It follows from (2.1)–(2.4) that $C(t,s)$ is an element of $L^2([0,1]^2, \mathbb{R})$; see Appendix A.2 in Horváth et al. (2013).

Remark 2.1. Assumption (2.4), which represents a weak dependence condition using a physical dependence measure, may be viewed as a substitute for classical mixing conditions. Mixing conditions in function spaces have been successfully employed to study consistency and asypmptotic normality in the presence of dependence in numerous settings, for example in nonparametric regression; see Masry (2005), Ferraty et al. (2002), and Ling et al. (2015). The relationship between classical mixing conditions and conditions along the lines of (2.4) in function spaces is discussed in Hörmann & Kokoszka (2010). In addition to the known pathologies of mixing conditions, in the context of the second order properties of spectral density estimates, they are typically accompanied by cummulant summability conditions, which we do not require here.

Due to the involved forms of the limits of partial sums and their functionals, starting with Politis & Romano (1994), resampling methods have played an important role in the applicability of functional time series. For some recent results on bootstrap in Hilbert spaces, we refer to Dehling et al. (2014, 2015). For an example of functional time series that do not satisfy such weak dependence conditions, see Benhenni et al. (2011), where functional long memory processes are developed.

We assume that the kernel K in the definition of $\hat{C}_N$ satisfies the following standard conditions:

$$K(0) = 1, \tag{2.5}$$

$$K \text{ is symmetric around } 0, \quad K(u) = 0 \text{ if } u > c \text{ with some } c > 0, \tag{2.6}$$

and

$$K \text{ is Lipschitz continuous on } [-c, c], \text{ where } c \text{ is given in } (2.6). \qquad (2.7)$$

Lastly we take the window (or smoothing parameter) h to satisfy that

$$h = h(N) \to \infty \quad \text{and} \quad \frac{h(N)}{N} \to 0, \quad \text{as} \quad N \to \infty. \qquad (2.8)$$

The main result of our paper establishes the asymptotic limit distribution of

$$Z_N(t, s) = \hat{C}_N(t, s) - \mathrm{E}\hat{C}_N(t, s).$$

Theorem 2.1. *If* (2.1)–(2.8) *and*

$$h/N^{\delta/(4+2\delta)} \to 0 \qquad (2.9)$$

hold, where δ is defined by (2.3), *then one can define a sequence of Gaussian processes $\Gamma_N(t, s)$ defined on the same probability space, and satisfying $\mathrm{E}\Gamma_N(t, s) = 0$, $\mathrm{E}\Gamma_N(t, s)\Gamma_N(t', s') = L(t, s, t', s')$ with*

$$L(t, s, t', s') = [C(t, s)C(t', s') + C(t, t')C(s, s')] \int_{-c}^{c} K^2(z)dz \qquad (2.10)$$

such that

$$\|(N/h)^{1/2}Z_N - \Gamma_N\| \xrightarrow{P} 0, \quad \text{as} \quad N \to \infty. \qquad (2.11)$$

Theorem 2.1 provides a Skorokhod–Dudley–Wichura representation of the weak convergence of Z_N to a Gaussian process. We note that the distribution of the limiting Gaussian process Γ_N does not depend on N, and hence the approximation (2.11) can be readily used to compute the limiting behavior of functionals of Z_N.

Assuming a higher moment condition than (2.3), a more optimal result can be proven in the sense that we can relax (2.9) in Theorem 2.1.

Theorem 2.2. *If* (2.1)–(2.8) *and*

$$\mathrm{E}\|X_0\|^8 < \infty \qquad (2.12)$$

are satisfied, then (2.11) *holds.*

Remark 2.2. Panaretos & Tavakoli (2013) and Hörmann et al. (2013) consider the estimation and, in the case of Panaretos & Tavakoli (2013), the asymptotic theory of more general objects that they refer to as the spectral density functions; they are defined by

$$f_\omega(t, s) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} exp(-\mathbf{i}\omega j)\gamma_j(t, s), \quad \omega \in [0, 2\pi),$$

where $\mathbf{i}$ is the imaginary unit. For a fixed ω, f_ω is estimated analogously to the long run covariance function by

$$\hat{f}_\omega(t, s) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} K\left(\frac{j}{h}\right) exp(-\mathbf{i}\omega j)\hat{\gamma}_j(t, s).$$

In this paper we only consider the asymptotics of $2\pi \hat{f}_0(t, s)$, but we could extend our results to the case of the joint asymptotics of $\hat{f}_\omega$ over ω as in Panaretos & Tavakoli (2013). This would require working with the Fourier transform component of the definition and follows along the lines of the univariate case as demonstrated in Brillinger (2001).

2.2. Bias, bandwidth selection, and positive definiteness

In order to infer from this the limit behavior of $\hat{C}_N - C$, we must also consider the bias. Following Parzen (1957), we assume that

$$\text{there exists a } \mathfrak{q} > 0 \text{ such that } 0 < \lim_{x \to 0} \frac{K(x) - 1}{|x|^{\mathfrak{q}}} = \mathfrak{K} < \infty, \tag{2.13}$$

and

$$\text{there exists a } \mathfrak{q}' > \mathfrak{q} \text{ such that } \sum_{\ell=-\infty}^{\infty} |\ell|^{\mathfrak{q}'} \|\gamma_\ell\| < \infty. \tag{2.14}$$

The asymptotic bias is given by $h^{-\mathfrak{q}} \mathfrak{F}(t, s)$, where

$$\mathfrak{F}(t, s) = \mathfrak{K} \sum_{\ell=-\infty}^{\infty} |\ell|^{\mathfrak{q}} \gamma_\ell(t, s).$$

Theorem 2.3. *If* (2.13), (2.14) *hold and* $h^{\mathfrak{q}}/N \to 0$, *then we have*

$$\left\| \mathrm{E}\hat{C}_N - C - h^{-\mathfrak{q}} \mathfrak{F} \right\| = o(h^{-\mathfrak{q}}).$$

We note that if the unbiased estimators $N\hat{\gamma}_{i,N}(t, s)/(N - i)$ are used in the definition of $\hat{C}_N$, then Theorem 2.3 remains true without assuming $h^{\mathfrak{q}}/N \to 0$.

The minimization of the asymptotic mean squared error provides a popular choice for h in case of univariate data; see Parzen (1957) and Andrews (1991). In our case the "optimal" h minimizes $\mathrm{E}\|\hat{C}_N - C\|^2$. Our results show that

$$\mathrm{E}\|\hat{C}_N - C\|^2 \approx \frac{h}{N} \mathrm{E}\|\Gamma_1\|^2 + h^{-2\mathfrak{q}} \|\mathfrak{F}\|^2. \tag{2.15}$$

Since

$$E\|\Gamma_1\|^2 = \left(\iint C^2(t, s) dt ds + \left(\int_0^1 C(t, t) dt \right)^2 \right) \int_{-c}^c K^2(u) du$$

we get that the minimum of the asymptotic value of the mean squared error in (2.15) is reached at

$$h_{opt} \approx \mathfrak{c}_0 N^{1/(1+2\mathfrak{q})},$$

where

$$\mathfrak{c}_0 = \left(\mathfrak{q}\|\mathfrak{F}\|^2 \right)^{1/(1+2\mathfrak{q})} \left(\left(\iint C^2(t, s) dt ds + \left(\int_0^1 C(t, t) dt \right)^2 \right) \int_{-c}^c K^2(u) du \right)^{-1/(1+2\mathfrak{q})}$$

6

The constant $\mathfrak{c}_0$ is a complicated function of the unknown correlations $\gamma_\ell(t,s)$ and the long run covariance function $C(t,s)$.

It may be possible to adapt the automatic smoothing parameter selection techniques developed for nonparametric regression in Rachdi & Vieu (2007) and Chagny & Roche (2014) in order to achieve a data driven minimizer of the expected value in (2.15). Another approach is to minimize (2.15) with respect to h after pilot estimators are chosen for $E\|\Gamma_1\|^2$ and $\|\mathfrak{F}\|^2$; see Bühlmann (1996) for a development in the univariate case. A data driven estimator is discussed in Horváth et al. (2014) for the "flat top" kernel, i.e. when $\mathfrak{q} = \infty$. Data driven bandwidth selection with functional time series remains lightly studied, and the authors plan on utilizing the second order properties of $\hat{C}_N$ to address this deficit in future work.

According to (2.15) and since h tends to infinity with N, the asymptotic integrated mean squared error is minimized by using a kernel K for which $\mathfrak{q}$ may be taken to be as large as possible. This encourages the use of a kernel function that is smooth or "flat" near the origin, but for arbitrary kernels $\hat{C}$ need not be positive definite. Several methods have been proposed to address related issues in the finite dimensional setting, see Politis (2011); they typically involve either sacrificing possible improvements in the bias by using a kernel that makes the estimator positive definite from the outset, like the Bartlett kernel, or using a higher order kernel and then altering the estimator to be positive definite by removing the negative eigenvalues from the diagonalization of the operator.

3. Application to the limit distribution of functional principal components

3.1. Asymptotic normality of eigenvalues and eigenfunctions

A technique to reduce the dimension of functional data that has received considerable attention, both in applications and theoretical investigations, is principal component analysis (PCA); we refer to Ramsay & Silverman (2005) and Horváth & Kokoszka (2012) for reviews of the subject. Typically the principal components used are computed as the eigenfunctions of the sample covariance function

$$\hat{C}_N^s(t,s) = \frac{1}{N} \sum_{i=1}^N (X_i(t) - \bar{X}_N(t))(X_i(s) - \bar{X}_N(s)).$$

Due to their important role in PCA, the difference between the empirical and theoretical eigenvalues and eigenfunctions have been investigated by several authors. Kokoszka & Horváth (2012, pp. 31–35) contains inequalities for the accuracy of the replacement of the theoretical PCA's with their empirical counterparts. The asymptotic normality of the deviation between the empirical and theoretical eigenvalues and eigenfunctions was proven by Dauxois et al. (1982), Bosq (2000), and Hall & Hosseini–Nasab (2007) assuming that the X_i's are independent and identically distributed. In great generality, Mas & Menneteau (2003) show that the asymptotic properties of the empirical eigenvalues and eigenfunctions are automatically inherited from the asymptotic properties of their corresponding operators. Kokoszka & Reimherr (2012) investigated the asymptotic

properties of the eigenvalues and eigenfunctions of $\hat{C}_N^s$ when the observations are from a stationary functional time series.

In case of inference with dependent functional data, it may be preferable to use the theoretical principal components $\{v_i\}_{i \geq 1}$ defined by the the long run covariance operator,

$$\lambda_i v_i(t) = \int C(t,s)v_i(s)ds, \ \ 1 \leq i < \infty, \tag{3.1}$$

where we have used $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ to denote the ordered eigenvalues. These define an example of dynamic functional principal components as defined in Hormann et al. (2014). The theoretical eigenvalues and eigenfunctions defined in (3.1) can be estimated from a sample by the eigenvalues and eigenfunctions of the empirical long run covariance function

$$\hat{\lambda}_i \hat{v}_i(t) = \int \hat{C}_N(t,s)\hat{v}_i(s)ds, \ \ 1 \leq i \leq N. \tag{3.2}$$

It was shown in Horváth et al. (2012) that if for some $p \geq 1$,

$$\lambda_1 > \ldots > \lambda_p > \lambda_{p+1} \geq 0, \tag{3.3}$$

then the estimators defined in (3.2) are asymptotically consistent in the sense that

$$\max_{1 \leq i \leq p} |\hat{\lambda}_i - \lambda_i| = o_P(1), \ \ \text{and} \ \ \max_{1 \leq i \leq p} \|\hat{s}_i \hat{v}_i - v_i\| = o_P(1), \ \ \ \text{as} \ \ N \to \infty,$$

where $\hat{s}_i = \text{sign}(\langle \hat{v}_i, v_i \rangle)$. We show that Theorems 2.1–2.3 imply the limit distributions of $(N/h)^{1/2}(\hat{\lambda}_i - \lambda_i)$, $(N/h)^{1/2}(\hat{v}_i(t) - v_i(t))$ and $(N/h)^{1/2}\|\hat{v}_i - v_i\|$, $1 \leq i \leq p$.

Theorem 3.1. *Under the conditions of Theorems 2.1 and 2.3, (3.3) and assuming* $\lim_{N \to \infty} N/h^{1+2q} = \mathfrak{a}$*, there exist random variables* $\mathfrak{g}_{\ell,N}$ *and random functions* $\mathcal{G}_{\ell,N}(t), 1 \leq \ell \leq p$ *such that*

$$\max_{1 \leq \ell \leq p} |(N/h)^{1/2}(\hat{\lambda}_\ell - \lambda_\ell) - \mathfrak{g}_{\ell,N}| = o_P(1),$$

$$\max_{1 \leq \ell \leq p} \|(N/h)^{1/2}(\hat{s}_\ell \hat{v}_\ell - v_\ell) - \mathcal{G}_{\ell,N}\| = o_P(1)$$

and

$$\{\mathfrak{g}_{\ell,N}, \mathcal{G}_{\ell,N}(t), 1 \leq \ell \leq p\}$$
$$\overset{\mathcal{D}}{=} \left\{ \lambda_\ell \left(2\int_{-c}^c K^2(z)dz\right)^{1/2} \mathcal{N}_{\ell,\ell} + \mathfrak{a} \iint \mathfrak{F}(u,s)v_\ell(s)v_\ell(u)duds, \right.$$
$$\left(\int_{-c}^c K^2(z)dz\right)^{1/2} \sum_{1 \leq k \neq \ell < \infty} v_k(t)\frac{(\lambda_\ell \lambda_k)^{1/2}}{\lambda_\ell - \lambda_k}\mathcal{N}_{\ell,k}$$
$$\left. + \mathfrak{a} \sum_{1 \leq k \neq \ell < \infty} \frac{v_k(t)}{\lambda_\ell - \lambda_k} \iint \mathfrak{F}(u,s)v_\ell(u)v_\ell(s)duds, 1 \leq \ell \leq p \right\},$$

where $\mathcal{N}_{\ell,k}$ *are independent standard normal random variables.*

If $\mathfrak{a} = 0$, then Theorem 3.1 implies that the empirical eigenvalues and eigenfunctions are asymptotically consistent with rate $(h/N)^{1/2}$. Theorem 3.1 also shows that $(N/h)^{1/2}(\hat{\lambda}_\ell - \lambda_\ell)$ are asymptotically independent and normally distributed, and that, on top of being orthogonal functions, $(N/h)^{1/2}(\hat{s}_\ell \hat{v}_k - v_k)$ $1 \le k \le p$ are asymptotically stochastically independent and Gaussian. This result is along the lines of the asymptotic independence and normality of the suitably normed and centered empirical eigenvalues and eigenfunctions of the empirical covariance function of independent and identically distributed functional observations. The main difference is the norming; we use $(N/h)^{1/2}$ in the case of the kernel estimator for the long run covariance function instead of the $N^{1/2}$ rate in the case of the sample covariance. Since $(N/h)^{1/2}\|\hat{s}_\ell \hat{v}_\ell - v_\ell\|^2, 1 \le \ell \le p$ are asymptotically independent, assuming that $\mathfrak{a} = 0$, Theorem 3.1 yields

$$\frac{N}{h}\|\hat{s}_\ell \hat{v}_\ell - v_\ell\|^2 \quad \overset{\mathcal{D}}{\to} \quad \lambda_\ell \int_{-c}^{c} K^2(z)dz \sum_{k \neq \ell} \frac{\lambda_k}{(\lambda_\ell - \lambda_k)^2} \mathcal{N}_{\ell,k}^2.$$

This is the analogue of the result of Dauxois et al. (1982) in the functional time series case for the long run covariance.

3.2. Statistical applications of Theorem 3.1

A fundamental issue in principal component analysis is the determination of an appropriate number of basis functions for dimension reduction; see Bande et al. (2015) and Kokoszka & Young (2015) for recent applications where this issue is faced. Theorem 3.1 supplies a natural procedure to determine this number, since it can be used to derive a test of whether a specified eigenvalue is different from zero. Specifically, this theory can be used to provide the null distribution of statistics similar to those proposed in Chapter 11.7.2 in Anderson (1984). Additionally, Theorem 3.1 could be used to test if principal components, or eigenvalues, based on C remain constant throughout the sample.

Another potential application of Theorem 3.1 is in assessing the accuracy of seminorms defined for the purpose of nonparametric functional regression with functional time series as presented in Chapter 3 of Ferraty & Vieu (2006). We may define a seminorm $\|\cdot\|_q^{PCA}$ by

$$\|f\|_q^{PCA} = \left(\sum_{i=1}^{q} \left(\int f(t) v_k(t) dt \right)^2 \right)^{1/2},$$

where the $v_k's$ are the eigenfunctions of the long run covariance operator. Theorem 3.1 may be used to provide an approximate distribution of for the Hilbert–Schmidt norm of estimates of these seminorms when the theoretical eigenvalues are replaced with their empirical versions. We leave these ideas as potential avenues of future research.

4. Proofs of Theorems 2.1–3.1

The proofs of the main results of the paper, Theorems 2.1 and Theorems 2.2, are carried out in three primary steps. Firstly, we show that the process Z_N can be well approximated by an analogous process $Z_{N,m}$ that is constructed using m–dependent random

functions with the aid of (2.4) in Subsection 4.1. Once we have achieved this approximation, we obtain a lower dimensional approximation $Z_{N,m}^d$ based on d dimensional random functions via a projection technique in Subsection 4.2. It is then straightforward to create a Gaussian approximation for this process (Subsection 4.3), and we may then retrace our steps with the Gaussian process by letting d and m tend to infinity (Subsection 4.4). We then obtain as a simple corollary the asymptotic distributions of the eigenvalues and eigenfunctions in Subsection 4.5.

4.1. Approximation with m–dependent sequences

To simplify notation, we assume throughout the proofs that $c = 1$ in (2.6) and (2.7). First we show that replacing the sample mean $\bar{X}_N(t)$ with $EX_0(t)$ in the definition of $\hat{\gamma}_i$ does not effect the limit distribution of Z_N^2. It is clear that we can assume without loss of generality that

$$EX_i(t) = 0. \tag{4.1}$$

Let

$$\tilde{C}_N(t,s) = \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \tilde{\gamma}_i(t,s),$$

where

$$\tilde{\gamma}_i(t,s) = \tilde{\gamma}_{i,N}(t,s) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{j=1}^{N-i} X_j(t) X_{j+i}(s), & i \geq 0 \\[4mm] \dfrac{1}{N} \displaystyle\sum_{j=1-i}^{N} X_j(t) X_{j+i}(s), & i < 0. \end{cases} \tag{4.2}$$

We prove in the following lemma that Z_N and $\tilde{Z}_N$ have the same limit distribution, where $\tilde{Z}_N(t,s) = \tilde{C}_N(t,s) - E\tilde{C}_N(t,s)$.

Lemma 4.1. *If* (2.1)–(2.8) *are satisfied, then we have that*

$$\frac{N}{h} \|Z_N - \tilde{Z}_N\|^2 = o_P(1). \tag{4.3}$$

Proof. It is easy to see that

$$\|Z_N - \tilde{Z}_N\| \tag{4.4}$$

$$\leq \|\bar{X}_N\| \left\{ \left\| \sum_{i=0}^{\infty} K\left(\frac{i}{h}\right) \frac{1}{N} \sum_{j=1}^{N-i} X_j \right\| + \left\| \sum_{i=0}^{\infty} K\left(\frac{i}{h}\right) \frac{1}{N} \sum_{j=i+1}^{N} X_j \right\| \right\}$$

$$+ \|\bar{X}_N\| \left\{ \left\| \sum_{i=-\infty}^{0} K\left(\frac{i}{h}\right) \frac{1}{N} \sum_{j=1-i}^{N} X_j \right\| + \left\| \sum_{i=-\infty}^{0} K\left(\frac{i}{h}\right) \frac{1}{N} \sum_{j=1}^{N+i} X_j \right\| \right\}$$

$$+ \|\bar{X}_N\|^2 \left| \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \right|.$$

Berkes et al. (2013) showed that under (2.1)–(2.4)

$$\|\bar{X}_N\| = O_P(N^{-1/2}), \tag{4.5}$$

and therefore by (2.6) and (2.7)

$$\|\bar{X}_N\|^2 \left| \sum_{i=-\infty}^{\infty} K\left(i/h\right) \right| = O_P(h/N).$$

On account of $EX_0(t)X_{i,i}(s) = 0$, by (2.4) we have that

$$\sum_{i=1}^{\infty} \left| \iint EX_0(t)X_i(s)dtds \right| \leq (E\|X_0\|^2)^{1/2} \sum_{i=1}^{\infty} (E\|X_0 - X_{0,i}\|^2)^{1/2} < \infty, \tag{4.6}$$

and therefore we obtain immediately that

$$\left\| \sum_{i=0}^{\infty} K\left(i/h\right) \frac{1}{N} \sum_{j=1}^{N-i} X_j \right\|^2 = N^{-2} \sum_{i,\ell=0}^{\infty} K\left(i/h\right) K\left(\ell/h\right) \sum_{j=1}^{N-i} \sum_{k=1}^{N-\ell} \iint EX_j(t)X_k(s)dtds$$

$$= O(1/N) \sum_{i,\ell=0}^{\infty} |K\left(i/h\right) K\left(\ell/h\right)|$$

$$= O_P(h^2/N),$$

where we used again (2.6) and (2.7). Thus we get by (4.5) that

$$\|\bar{X}_N\| \left\| \sum_{i=0}^{\infty} K\left(i/h\right) \frac{1}{N} \sum_{j=1}^{N-i} X_j \right\| = O_P(h/N).$$

Similar arguments provide the same upper bounds for the other terms in (4.4) which implies that $(N/h)^{1/2}\|\tilde{Z}_N - Z_N\| = O_P((h/N)^{1/2}) = o_P(1)$ which gives (4.3) . $\qquad\square$

We recall $X_{n,m}, 1 \leq n \leq N$ defined in (2.4) for $m \geq 0$. Replacing X_i with $X_{i,m}$ in the definition of $\tilde{C}_N$ we define

$$\tilde{C}_{N,m}(t, s) = \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \tilde{\gamma}_i^{(m)}(t, s),$$

where

$$\tilde{\gamma}_i^{(m)}(t, s) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{j=1}^{N-i} X_{j,m}(t)X_{j+i,m}(s), & i \geq 0 \\[2ex] \dfrac{1}{N} \displaystyle\sum_{j=1-i}^{N} X_{j,m}(t)X_{j+i,m}(s), & i < 0 \end{cases}$$

and

$$\tilde{Z}_{N,m}(t, s) = \tilde{C}_{N,m}(t, s) - E\tilde{C}_{N,m}(t, s).$$

11

Lemma 4.2. *If* (2.1)–(2.8) *are satisfied, then we have*

$$\lim_{m \to \infty} \|C_m - C\| = 0 \tag{4.7}$$

where

$$C_m(t, s) = \sum_{\ell=-m}^{m} \mathrm{E}X_{0,m}(t)X_{\ell,m}(s).$$

Also, as $m \to \infty$,

$$\int C_m(t, t)dt \to \int C(t, t)dt, \tag{4.8}$$

$$\iint \left(\sum_{\ell=-\infty}^{\infty} \mathrm{E}X_0(t)X_{\ell,m}(s) \right) \left(\sum_{j=-\infty}^{\infty} \mathrm{E}X_{0,m}(t)X_\ell(s) \right) dtds \to \|C\|^2, \tag{4.9}$$

and

$$\int \sum_{\ell=-\infty}^{\infty} \mathrm{E}X_0(t)X_{\ell,m}(t)dt \to \int C(t, t)dt, \tag{4.10}$$

Proof. By definition we have

$$C(t, s) = \sum_{\ell=-\infty}^{-m-1} \mathrm{E}X_0(t)X_\ell(s) + \sum_{\ell=m+1}^{\infty} \mathrm{E}X_0(t)X_\ell(s) + \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)X_\ell(s).$$

Due to the fact that $C(t, s)$ is in $L^2([0, 1]^2, \mathbb{R})$, it follows that

$$\left\| \sum_{\ell=m+1}^{\infty} \mathrm{E}X_0(t)X_\ell(s) \right\| \to 0, \quad \text{as} \quad m \to \infty$$

and

$$\left\| \sum_{\ell=-\infty}^{-m-1} \mathrm{E}X_0(t)X_\ell(s) \right\| \to \infty, \quad \text{as} \quad m \to \infty.$$

Clearly,

$$\mathrm{E}X_0(t)X_\ell(s) - \mathrm{E}X_{0,m}(t)X_{\ell,m}(s)$$
$$= \mathrm{E}X_0(t)X_\ell(s) - \mathrm{E}X_{0,m}(t)X_\ell(s) + \mathrm{E}X_{0,m}(t)X_\ell(s) - \mathrm{E}X_{0,m}(t)X_{\ell,m}(s)$$

and therefore by (2.4) and stationarity we conclude

$$\|\mathrm{E}X_0(t)X_\ell(s) - \mathrm{E}X_{0,m}(t)X_{\ell,m}(s)\| \le 2(\mathrm{E}\|X_0\|^2\mathrm{E}\|X_0 - X_{0,m}\|^2)^{1/2}.$$

Hence

$$\left\| \sum_{\ell=-m}^{m} (\mathrm{E}X_0(t)X_\ell(s) - \mathrm{E}X_{0,m}(t)X_{\ell,m}(s)) \right\| \le 2(2m + 1)(\mathrm{E}\|X_0\|^2\mathrm{E}\|X_0 - X_{0,m}\|^2)^{1/2} \to 0,$$

12

as $m \to \infty$, completing the proof of (4.7). Similar arguments give (4.8).
To prove (4.9) we first define

$$r_{1,m} = \{|\ell| > m, |j| \le m\}, \quad r_{2,m} = \{|\ell| \le m, |j| > m\}, \quad r_{3,m} = \{|\ell| > m, |j| > m\}$$

and

$$r_{4,m} = \{|\ell|, |j| \le m\}.$$

Let

$$\alpha_{\ell,j,m}(t,s) = \mathrm{E}X_0(t)X_{\ell,m}(s)\mathrm{E}X_{0,m}(t)X_j(s) - a_\ell(t,s)a_j(t,s), \text{ where } a_\ell(t,s) = \mathrm{E}X_0(t)X_\ell(s).$$

For all $\ell > m$ we have that $\mathrm{E}X_0(t)X_{\ell,m}(s) = 0$ and therefore by the Cauchy–Schwarz inequality

$$\left| \iint \sum_{r_{1,m,1}} \alpha_{\ell,j,m}(t,s) dt ds \right| = \left| \iint \sum_{r_{1,m,1}} a_\ell(t,s)a_j(t,s) dt ds \right|$$

$$\le 2\mathrm{E}\|X_0\|^2 \sum_{\ell>m}^{\infty} (\mathrm{E}\|X_0 - X_{0,\ell}\|^2)^{1/2} \sum_{j=0}^{\infty} (\mathrm{E}\|X_0 - X_{0,j}\|^2)^{1/2}$$

$$\to 0, \quad \text{as } m \to \infty,$$

where $r_{1,m,1} = \{\ell > m, |j| \le m\}$. On the set $r_{1,m,2} = \{\ell < -m, |j| \le m\}$ we write by the independence of $X_{0,\ell}$ and $X_{\ell,m}$ that

$$\sum_{r_{1,m,2}} \iint |\mathrm{E}X_0(t)X_{\ell,m}(s)\mathrm{E}X_{0,m}(t)X_j(s)| dt ds$$

$$= \sum_{r_{1,m,2}} \iint |\mathrm{E}(X_{0,m}(t) - X_{-\ell}(t))X_{\ell,m}(s)\mathrm{E}X_{0,m}(t)X_j(s)| dt ds$$

$$\le \sum_{r_{1,m,2}} (\mathrm{E}\|X_0 - X_{0,-\ell}\|^2)^{1/2}(\mathrm{E}\|X_0\|^2)^{3/2}$$

$$\le (2m+1)(\mathrm{E}\|X_0\|^2)^{3/2} \sum_{\ell=m}^{\infty} (\mathrm{E}\|X_0 - X_{0,\ell}\|^2)^{1/2}.$$

It follows similarly that

$$\iint \sum_{r_{1,m,2}} |a_\ell(t,s)a_j(t,s)| dt ds \le (2m+1)(\mathrm{E}\|X_0\|^2)^{3/2} \sum_{\ell=m}^{\infty} (\mathrm{E}\|X_0 - X_{0,\ell}\|^2)^{1/2}$$

resulting in

$$\left| \iint \sum_{r_{1,m}} \alpha_{\ell,j,m}(t,s) dt ds \right| \to 0, \quad \text{as } m \to \infty$$

via (2.4). Similar arguments give for $i = 2, 3, 4$ that

$$\left| \iint_{r_{i,m}} \alpha_{\ell,j,m}(t,s) dt ds \right| \to 0, \quad \text{as } m \to \infty.$$

13

Observing that

$$\iint \sum_{\ell,i=-\infty}^{\infty} a_\ell(t,s)a_i(t,s)dtds = \|C\|^2,$$

the proof of (4.9) is complete. The proof of (4.10) goes along the lines of (4.9). □

Lemma 4.3. *If (2.1)–(2.8) are satisfied, then we have*

$$\lim_{m\to\infty}\limsup_{N\to\infty}\frac{N}{h}\mathrm{E}\iint(\tilde{Z}_N(t,s)-\tilde{Z}_{N,m}(t,s))^2dtds = 0. \tag{4.11}$$

Also, for each $m \geq 1$

$$\lim_{N\to\infty}\frac{N}{h}\iint \mathrm{var}(\tilde{Z}_N(t,s))dtds = \left(\|C\|^2 + \left(\int C(t,t)dt\right)^2\right)\int_{-1}^{1}K^2(u)du, \tag{4.12}$$

$$\lim_{N\to\infty}\frac{N}{h}\iint \mathrm{var}(\tilde{Z}_{N,m}(t,s))dtds = \left(\|C_m\|^2 + \left(\int C_m(t,t)dt\right)^2\right)\int_{-1}^{1}K^2(u)du, \tag{4.13}$$

and

$$\lim_{N\to\infty}\frac{N}{h}\iint \mathrm{cov}(\tilde{Z}_N(t,s),\tilde{Z}_{N,m}(t,s))dtds \tag{4.14}$$

$$= \left\{\iint\left(\sum_{\ell=-\infty}^{\infty}\mathrm{E}X_0(t)X_{\ell,m}(s)\right)\left(\sum_{j=-\infty}^{\infty}\mathrm{E}X_{0,m}(t)X_j(s)\right)dtds\right.$$

$$\left. + \left(\int\sum_{\ell=-\infty}^{\infty}\mathrm{E}X_0(t)X_{\ell,m}(t)dt\right)^2\right\}\int_{-1}^{1}K^2(u)du.$$

Proof. By a simple calculation

$$\frac{N}{h}\mathrm{E}\iint\left(\tilde{Z}_N(t,s)-\tilde{Z}_{N,m}(t,s)\right)^2dtds$$

$$= \frac{N}{h}\iint\mathrm{var}(\tilde{Z}_N(t,s))dtds + \frac{N}{h}\iint\mathrm{var}(\tilde{Z}_{N,m}(t,s))dtds$$

$$- 2\frac{N}{h}\iint\mathrm{cov}(\tilde{Z}_N(t,s),\tilde{Z}_{N,m}(t,s))dtds,$$

and hence (4.11) follows from Lemma 4.2 and (4.12)–(4.14).
We recall $a_\ell(t,s) = \mathrm{E}X_0(t)X_\ell(s)$ and let

$$\psi_{\ell,r,p}(t,s) = \mathrm{E}[X_0(t)X_\ell(s)X_r(t)X_p(s)]$$
$$- a_\ell(t,s)a_{p-r}(t,s) - a_r(t,t)a_{p-\ell}(s,s) - a_p(t,s)a_{r-\ell}(t,s).$$

As the first step in the proof of (4.12) we show that

$$\frac{1}{h}\sum_{g,\ell=-h}^{h}\sum_{r=-(N-1)}^{N-1}\left|\iint\psi_{\ell,r,r+g}(t,s)dtds\right| \to 0, \quad \text{as } N \to \infty. \tag{4.15}$$

14

It is easy to see that

$$\frac{1}{h}\sum_{g,\ell=-h}^{h}\sum_{r=-(N-1)}^{N-1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right| \tag{4.16}$$

$$=\frac{1}{h}\sum_{\ell=0}^{h}\sum_{g=0}^{h}\sum_{r=0}^{N-1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|$$

$$+\frac{1}{h}\sum_{\ell=0}^{h}\sum_{g=0}^{h}\sum_{r=-(N-1)}^{-1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|$$

$$+\cdots+\frac{1}{h}\sum_{\ell=-h}^{-1}\sum_{g=-h}^{-1}\sum_{r=-(N-1)}^{-1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|,$$

where the right hand side contains eight terms corresponding to the combinations of the indices ℓ, g and r taking either nonnegative or negative values. Due to stationarity, we only consider the first term. In the summation of $\psi_{\ell,r,r+g}$, we consider three cases: ℓ is less than r, ℓ is between r and $r+g$, or ℓ is larger than $r+g$.
Let $R_1 = \{(\ell,g,r) : \ell < r, 0 \le \ell, g \le h, 0 \le r \le N-1\}, R_2 = \{(\ell,g,r) : r \le \ell \le r+g, 0 \le \ell, g \le h, 0 \le r \le N-1\}$, and $R_3 = \{(\ell,g,r) : r+g < \ell, 0 \le \ell, g \le h, 0 \le r \le N-1\}$. Clearly,

$$\sum_{\ell=0}^{h}\sum_{g=0}^{h}\sum_{r=0}^{N-1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right| \le U_{1,N} + U_{2,N} + U_{3,N},$$

where

$$U_{1,N} = \sum_{R_1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|, \quad U_{2,N} = \sum_{R_2}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|,$$

and

$$U_{3,N} = \sum_{R_3}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right|.$$

Using the definition of $\psi_{\ell,r,r+g}$ we write

$$\sum_{R_1}\left|\iint \psi_{\ell,r,r+g}(t,s)dtds\right| \tag{4.17}$$

$$\le \sum_{R_1}\left|\iint a_r(t,t)a_{r+g-\ell}(s,s)dtds\right|$$

$$+\sum_{R_1}\left|\iint a_{r-\ell}(t,s)a_{r+g}(t,s)dtds\right|$$

$$+\sum_{R_1}\left|\iint [\mathrm{E}X_0(t)X_\ell(s)X_r(t)X_{r+g}(s) - a_\ell(t,s)a_g(t,s)]dtds\right|.$$

15

By the inequality (A.3) in Horváth & Rice (2014) and the fact that for any random variable $(\mathrm{E}\zeta^2)^{1/2} \leq (\mathrm{E}\zeta^4)^{1/4}$ we get that

$$\left| \iint a_r(t,t)a_{r+g-\ell}(s,s)dtds \right| \leq \mathrm{E}\|X_0\|^2 c_r c_{r+g-\ell} \tag{4.18}$$

and

$$\left| \iint a_{r-\ell}(t,s)a_{r+g}(t,s)dtds \right| \leq \mathrm{E}\|X_0\|^2 c_{r-\ell} c_{r+g}, \tag{4.19}$$

where, we recall from (2.4),

$$c_\ell = (\mathrm{E}\|X_0 - X_{0,\ell}\|^4)^{1/4}.$$

Combining (4.18) with the definition of R_1 we conclude

$$\sum_{R_1} \left| \iint a_r(t,t)a_{r+g-\ell}(s,s)dtds \right| \leq \mathrm{E}\|X_0\|^2 \sum_{R_1} c_r c_{r+g-\ell} \tag{4.20}$$

$$\leq \mathrm{E}\|X_0\|^2 \sum_{\ell=0}^{h} \sum_{r=\ell+1}^{N-1} c_r \sum_{g=0}^{h} c_{r+g-\ell}$$

$$\leq \mathrm{E}\|X_0\|^2 \left(\sum_{\ell=0}^{\infty} \sum_{r=\ell}^{\infty} c_r \right) \sum_{g=0}^{\infty} c_g.$$

Similarly,

$$\sum_{R_1} \left| \iint a_{r-\ell}(t,s)a_{r+g}(t,s)dtds \right| \leq \mathrm{E}\|X_0\|^2 \sum_{R_1} c_{r-\ell} c_{r+g} \tag{4.21}$$

$$\leq \mathrm{E}\|X_0\|^2 \sum_{\ell=0}^{h} \sum_{r=\ell}^{N-1} \sum_{g=0}^{h} c_{r-\ell} c_{r+g}$$

$$\leq \mathrm{E}\|X_0\|^2 \sum_{\ell=0}^{h} \sum_{r=\ell}^{N-1} \sum_{p=\ell}^{\infty} c_{r-\ell} c_p$$

$$\leq \mathrm{E}\|X_0\|^2 \left(\sum_{\ell=0}^{\infty} \sum_{p=\ell}^{\infty} c_p \right) \sum_{r=0}^{\infty} c_r.$$

Let $1 \leq \xi = \xi(N) \leq h$ be a sequence of real numbers which will be defined below. We write $R_1 = R_{1,1} \cup R_{1,2}$, where $R_{1,1} = \{(\ell, g, r) \in R_1 : r - \ell > \xi\}$ and $R_{1,2} = \{(\ell, g, r) \in R_1 : r - \ell \leq \xi\}$. It follows from (A.9) of Horváth & Rice (2014) that there is a constant A_1, depending only on the distribution of X_0 such that for all $(\ell, g, r) \in R_{1,1}$

$$\left| \iint [\mathrm{E}X_0(t)X_\ell(s)X_r(t)X_{r+g}(s) - a_\ell(t,s)a_g(t,s)]dtds \right| \leq A_1(c_{r-\ell} + c_{r+g-\ell}).$$

Thus we get that

$$\sum_{R_{1,1}} \left| \iint [\mathrm{E}X_0(t)X_\ell(s)X_r(t)X_{r+g}(s) - a_\ell(t,s)a_g(t,s)]dtds \right| \tag{4.22}$$

16

$$\leq A_1 \sum_{\ell=0}^{h} \sum_{r=\ell+\xi}^{N} \sum_{g=\ell-r}^{h} (c_{r-\ell} + c_{r+g-\ell})$$

$$\leq 2A_1 h^2 \sum_{p=\xi}^{\infty} c_p.$$

To obtain an upper bound when the summation is over $R_{1,2}$ we write $R_{1,2} = \cup_{i=1}^{3} R_{1,2,i}$ where $R_{1,2,1} = \{(\ell, g, r) \in R_{1,2} : \ell > \xi\}, R_{1,2,2} = \{(\ell, g, r) \in R_{1,2} : g > \xi\}$ and $R_{1,2,3} = \{(\ell, g, r) \in R_{1,2} : \ell, g \leq \xi\}$. It follows from (A.4) in Horváth & Rice (2014) that there is a constant A_2 depending only on X_0 such that

$$\left| \iint E X_0(t) X_\ell(s) X_r(t) X_{r+g}(s) dt ds \right| \leq A_2 \min(c_\ell, c_g) \quad \text{for all} \quad (\ell, g, r) \in R_1.$$

Thus we have

$$\sum_{R_{1,2,1}} \left| \iint E X_0(t) X_\ell(s) X_r(t) X_{r+g}(s) dt ds \right| \tag{4.23}$$

$$\leq A_2 \sum_{\ell=\xi+1}^{h} \sum_{r=\ell}^{\ell+\xi} \sum_{g=\ell-r}^{h} c_\ell$$

$$\leq A_2 h \xi \sum_{\ell=\xi}^{\infty} c_\ell.$$

Similarly,

$$\sum_{R_{1,2,2}} \left| \iint E X_0(t) X_\ell(s) X_r(t) X_{r+g}(s) dt ds \right| \leq A_2 h \xi \sum_{\ell=\xi}^{\infty} c_\ell. \tag{4.24}$$

It follows from the definitions of $R_{1,2,3}$ and $R_{1,2}$ that $R_{1,2,3} \subseteq \{0 \leq \ell, g \leq \xi, 0 \leq r \leq 2\xi\}$, so we have with some constant A_3 that

$$\sum_{R_{1,2,3}} \left| \iint E X_0(t) X_\ell(s) X_r(t) X_{r+g}(s) dt ds \right| \leq A_3 \xi^3. \tag{4.25}$$

Similar but somewhat easier arguments show

$$\sum_{R_{1,2}} \left| \iint a_\ell(t,s) a_g(t,s) dt ds \right| \leq A_4 \left\{ h\xi \sum_{\ell=\xi}^{\infty} c_\ell + \xi^3 \right\}. \tag{4.26}$$

with some constant A_4. If (2.4) holds, then $\sum_{i=1}^{\infty} c_i < \infty$ and

$$\ell w \left(\sum_{i=\ell}^{\infty} c_i \right) \to 0, \quad \text{as } \ell \to \infty. \tag{4.27}$$

Also $w^{-1}(x)$ exists for small enough x; see Bingham et al. (1987), pp. 28 and 29, and $w^{-1}(x)/x^3 \to \infty$ as $x \to 0$. Using theorem 1.5.12 of Bingham et al. (1987) we get that (4.27) is equivalent to

$$\frac{1}{w^{-1}(1/\ell)} \sum_{i=\ell}^{\infty} c_i \to 0, \quad \text{as } \ell \to \infty. \tag{4.28}$$

Therefore, with the choice of

$$\xi = \frac{1}{w(1/h)}$$

in (4.22)–(4.26) we obtain that

$$\frac{1}{h} \sum_{R_1} \left| \iint [EX_0(t)X_\ell(s)X_r(t)X_{r+g}(s) - a_\ell(t,s)a_g(t,s)]dtds \right| \to 0. \tag{4.29}$$

Putting together (4.20), (4.21) and (4.29) we conclude

$$\frac{1}{h} \sum_{R_1} \left| \iint \psi_{\ell,r,r+g}(t,s)dtds \right| \to 0. \tag{4.30}$$

Similar arguments show that (4.30) remains true if the domain of summation R_1 is replaced with R_2 or R_3 and hence

$$\frac{1}{h} \sum_{\ell=0}^{h} \sum_{g=0}^{h} \sum_{r=0}^{N-1} \left| \iint \psi_{\ell,r,r+g}(t,s)dtds \right| \to 0.$$

With minor modifications of the arguments above one can verify that the remaining seven terms in (4.16) also tend to 0, as $N \to \infty$.

Now we show that (4.15) implies (4.12). By a simple calculation using (2.1) and (4.2) we get

$$N\text{cov}(\tilde{\gamma}_\ell(t,s), \tilde{\gamma}_g(t,s))$$

$$= \frac{1}{N} \left\{ \sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)} \sum_{j=\max(1,1-g)}^{\min(N,N-g)} EX_i(t)X_{i+\ell}(s)X_j(t)X_{j+g}(s) \right.$$

$$\left. - (N - |\ell|)(N - |g|)a_\ell(t,s)a_g(t,s) \right\}$$

$$= \frac{1}{N} \sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)} \sum_{j=\max(1,1-g)}^{\min(N,N-g)} \left(\psi_{\ell,j-i,j-i+g}(t,s) \right.$$

$$\left. + a_{j-i+g}(t,s)a_{j-i-\ell}(t,s) + a_{j-i}(t,t)a_{j-i+g-\ell}(s,s) \right).$$

Notice that the summand in the last formula depends only on the difference $j - i$. Let $\varphi_N(r,\ell,g)$ denote the cardinality of the set $\{(i,j) : j - i = r, \max(1, 1 - \ell) \le i \le$

$\min(N, N - \ell), \max(1, 1 - g) \leq j \leq \min(N, N - g)\}$, i.e. $\varphi_N(r, \ell, g)$ is the number of pairs of indices i, j in the sum so that $j - i = r$. Clearly, $\varphi_N(r, \ell, g) \leq N$. Also, $\varphi_N(r, \ell, g) \geq N - 2(|\ell| + |r| + |g|)$, since $\{(i, i + r) : \max(|r|, 1 - \ell + |r|, 1 - g + |r|) \leq i \leq \min(N - |r|, N - g - |r|, N - \ell - |r|)\} \subseteq \{(i, j) : j - i = r, \max(1, 1 - \ell) \leq i \min(N, N - \ell), \max(1, 1 - g) \leq j \leq \min(N, N - g)\}$. Using the notation

$$\bar{\varphi}_N(r, \ell, g) = \varphi_N(r, \ell, g)/N \tag{4.31}$$

we can write

$$N\mathrm{cov}(\tilde{\gamma}_\ell(t, s), \tilde{\gamma}_g(t, s))$$
$$= \sum_{r=-(N-1)}^{N-1} \bar{\varphi}_N(r, \ell, g) \{\psi_{\ell, r, r+g}(t, s) + a_{r+g}(t, s)a_{r-\ell}(t, s) + a_r(t, t)a_{r+g-\ell}(s, s)\}.$$

It follows that

$$\frac{N}{h}\mathrm{var}(\tilde{C}_N(t, s)) = \frac{N}{h} \sum_{g, \ell=-h}^{h} K(g/h)K(\ell/h)\mathrm{cov}(\tilde{\gamma}_\ell(t, s), \tilde{\gamma}_g(t, s))$$
$$= q_{1,N}(t, s) + q_{2,N}(t, s) + q_{3,N}(t, s),$$

where

$$q_{1,N}(t, s) = \frac{1}{h} \sum_{g, \ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K(g/h)K(\ell/h)\bar{\varphi}_N(r, \ell, g)\psi_{\ell, r, r+g}(t, s),$$

$$q_{2,N}(t, s) = \frac{1}{h} \sum_{g, \ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K(g/h)K(\ell/h)\bar{\varphi}_N(r, \ell, g)a_{r+g}(t, s)a_{r-\ell}(t, s),$$

$$q_{3,N}(t, s) = \frac{1}{h} \sum_{g, \ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K(g/h)K(\ell/h)\bar{\varphi}_N(r, \ell, g)a_r(t, t)a_{r+g-\ell}(s, s).$$

We start with $q_{2,N}$. Let $\varepsilon > 0$. By a change of variables we have

$$q_{2,N}(t, s) = \frac{1}{h} \sum_{|u|, |v| \leq h+N-1} \sum_{r=b_1}^{b_2} K\left(\frac{u - r}{h}\right) K\left(\frac{v - r}{h}\right) \bar{\varphi}_N(r, r - v, u - r)a_u(t, s)a_v(t, s),$$

where

$$b_1 = b_1(u, v, N) = \max(u - h, v - h, -(N - 1)) \tag{4.32}$$

and

$$b_2 = b_2(u, v, N) = \min(u + h, v + h, N - 1). \tag{4.33}$$

If

$$q_{2,N}^{(M)}(t, s) = \frac{1}{h} \sum_{|u|, |v| \leq M} \sum_{r=b_1}^{b_2} K\left(\frac{u - r}{h}\right) K\left(\frac{v - r}{h}\right) \bar{\varphi}_N(r, r - v, u - r)a_u(t, s)a_v(t, s),$$

19

then we have

$$|q_{2,N}(t,s) - q_{2,N}^{(M)}(t,s)|$$

$$\leq \frac{1}{h} \sum_{u,v \in \Theta_{N,M}} \sum_{r=b_1}^{b_2} \left| K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) \bar{\varphi}_N(r, r-v, u-r) a_u(t,s) a_v(t,s) \right|,$$

where $\Theta_{N,M} = \{u, v : |u|, |v| \leq h + N - 1, \max(|u|, |v|) \geq M\}$. By assumption (2.6), the number of terms in r such that $b_1(u,v,N) \leq r \leq b_2(u,v,N)$ and $K((u-r)/h)K((v-r)/h) \neq 0$ cannot exceed $2h$ for any u, v. Since $|\bar{\varphi}_N| \leq 1$, we conclude

$$|q_{2,N}(t,s) - q_{2,N}^{(M)}(t,s)| \leq 2 \sup_{|x| \leq 1} K^2(x) \sum_{u,v \in \Theta_{N,M}} |a_u(t,s) a_v(t,s)|. \qquad (4.34)$$

The Cauchy-Schwarz inequality yields

$$\iint |q_{2,N}(t,s) - q_{2,N}^{(M)}(t,s)| \, dt ds \qquad (4.35)$$

$$\leq 2 \sup_{|x| \leq 1} K^2(x) \left[\left(\sum_{|u| \geq M} \|a_u\| \right)^2 + 2 \left(\sum_{|u| > M} \|a_u\| \right) \left(\sum_{v=-\infty}^{\infty} \|a_v\| \right) \right] < \varepsilon/4,$$

by taking M sufficiently large. We recall that $N - 2(r + \ell + g) \leq \varphi(r, \ell, g) \leq N$. If $|u|, |v| \leq M, b_1(u,v,N) \leq r \leq b_2(u,v,N)$ hold, then $|r| \leq M + h$ and hence for such u, v and r we also have $|\varphi(r, r-v, u-r) - N| \leq 2|r + r - v + u - r| \leq 2(|r| + |u| + |v|) \leq 2(h + 3M)$, resulting in that $|\bar{\varphi}_N(r, r-v, u-r) - 1| \leq 2(3M + h)/N$. Using (2.8), one can establish along the lines of the proof of (4.35)

$$\iint \left| q_{2,N}^{(M)}(t,s) - \frac{1}{h} \sum_{|u|,|v| \leq M} \sum_{r=b_1}^{b_2} K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) a_u(t,s) a_v(t,s) \right| dt ds \qquad (4.36)$$

$$< \varepsilon/4$$

for all large enough N. By (2.7) and (2.8), for any $\eta > 0$ we have

$$\sup_{|u|,|v| \leq M} \sup_{r} \left| K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) - K^2\left(\frac{r}{h}\right) \right| < \eta,$$

when N is sufficiently large. Since we can take $\eta > 0$ as small as we wish, it holds for all large enough N that

$$\frac{1}{h} \sum_{|u|,|v| \leq M} \sum_{r=b_1}^{b_2} \iint \left| \left[K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) - K^2\left(\frac{r}{h}\right) \right] a_u(t,s) a_v(t,s) \right| dt ds \qquad (4.37)$$

$$< \varepsilon/4.$$

Clearly, according to the definition of a Riemann integral

$$\sup_{|u|,|v| \leq M} \left| \frac{1}{h} \sum_{r=b_1(u,v,N)}^{b_2(u,v,N)} K^2\left(\frac{r}{h}\right) - \int_{-1}^{1} K^2(z) dz \right| \to 0, \quad \text{as } N \to \infty.$$

20

Using the definition of $C(t, s)$ one can easily see via the Lebesgue dominated convergence theorem that

$$\iint \sum_{|u|,|v| \le M} a_u(t,s) a_v(t,s) dt ds = \iint \left(\sum_{|u| \le M} a_u(t,s) \right)^2 dt ds \to \|C\|^2,$$

as $M \to \infty$. Thus we get that for all N and M sufficiently large

$$\left| \frac{1}{h} \iint \sum_{|u|,|v| \le M} \sum_{r=b_1}^{b_2} K^2 \left(\frac{r}{h} \right) a_u(t,s) a_v(t,s) dt ds - \|C\|^2 \int_{-1}^{1} K^2(z) dz \right| \le \varepsilon/4. \qquad (4.38)$$

Combining (4.34)–(4.38) we conclude

$$\lim_{N \to \infty} \iint q_{2,N}(t,s) dt ds = \|C\|^2 \int_{-1}^{1} K^2(z) dz. \qquad (4.39)$$

Observing that

$$\iint \sum_{|u|,|v| \le M} a_u(t,t) a_v(s,s) dt ds = \left(\int \sum_{|u| \le M} a_u(t,t) dt \right)^2 \to \left(\int C(t,t) dt \right)^2,$$

as $M \to \infty$, minor modifications of the proof of (4.39) yield

$$\lim_{N \to \infty} \iint q_{3,N}(t,s) dt ds = \left(\int C(t,t) dt \right)^2 \int_{-1}^{1} K^2(z) dz. \qquad (4.40)$$

Finally, by (4.15)

$$\left| \iint q_{1,N}(t,s) dt ds \right| \le \frac{1}{h} \sup_{|x| \le c} K^2(x) \sum_{g,\ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} \left| \iint \psi_{\ell,r,r+g}(t,s) dt ds \right| \to 0, \quad (4.41)$$

as $N \to \infty$. The result in (4.12) now follows from (4.39)–(4.41).
Clearly, (4.13) is a special case of (4.12).
Let

$$\psi_{\ell,r,p}^{(m)}(t,s) = EX_0(t) X_\ell(s) X_{r,m}(t) X_{p,m}(s) - a_\ell(t,s) a_{p-r,m}(t,s)$$
$$- a_{r,m}^{(2)}(t,t) a_{p-\ell,m}^{(2)}(s,s) - a_{p,m}^{(2)}(t,s) a_{r-\ell,m}^{(1)}(t,s),$$

where

$$a_{\ell,m}(t,s) = EX_{0,m} X_{\ell,m}(s), a_{\ell,m}^{(1)}(t,s) = EX_{0,m}(t) X_\ell(s) \text{ and } a_{\ell,m}^{(2)}(t,s) = EX_0(t) X_{\ell,m}(s).$$

Under the conditions of the Theorem 2.1 we have that

$$\frac{1}{h} \sum_{g,\ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} \left| \iint \psi_{\ell,r,r+g}^{(m)}(t,s) dt ds \right| \to 0, \quad as \quad N \to \infty, \qquad (4.42)$$

21

along the lines of (4.15). It follows from the definitions of $\tilde{Z}_N$ and $\tilde{Z}_{N,m}$ that

$$\frac{N}{h}\mathrm{E}\tilde{Z}_N(t,s)\tilde{Z}_{N,m}(t,s) = \frac{N}{h}\sum_{\ell=-h}^{h}\sum_{k=-h}^{h}K\left(\frac{\ell}{h}\right)K\left(\frac{k}{h}\right)\mathrm{cov}(\tilde{\gamma}_\ell(t,s),\tilde{\gamma}_{g,m}(t,s)).$$

Also,

$$N\mathrm{cov}(\tilde{\gamma}_\ell(t,s),\tilde{\gamma}_{g,m}(t,s))$$

$$= \frac{1}{N}\Bigg\{\sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)}\sum_{j=\max(1,1-g)}^{\min(N,N-g)}\psi_{\ell,j-i,j-i+g}^{(m)}(t,s)$$

$$+ a_{j-i+g,m}^{(2)}(t,s)a_{j-i-\ell,m}^{(1)}(t,s) + a_{j-i,m}^{(2)}(t,t)a_{j-i+g-\ell}^{(2)}(s,s)\Bigg\}.$$

Following the proof of (4.41) one can show that (4.42) implies

$$\frac{1}{h}\left|\iint\sum_{\ell=-h}^{h}\sum_{k=-h}^{h}\sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)}\sum_{j=\max(1,1-g)}^{\min(N,N-g)}K\left(\frac{\ell}{h}\right)K\left(\frac{k}{h}\right)\psi_{\ell,j-i,j-i+g}^{(m)}(t,s)dtds\right| \to 0,$$

as $N \to \infty$. Along the lines of (4.39) and (4.40) we get that

$$\frac{1}{h}\sum_{\ell=-h}^{h}\sum_{k=-h}^{h}\sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)}\sum_{j=\max(1,1-g)}^{\min(N,N-g)}K\left(\frac{\ell}{h}\right)K\left(\frac{k}{h}\right)\iint a_{j-i+g,m}^{(2)}(t,s)a_{j-i-\ell,m}^{(1)}(t,s)dtds$$

$$\to \iint\left(\sum_{\ell=-\infty}^{\infty}\mathrm{E}X_0(t)X_{\ell,m}(s)\right)\left(\sum_{j=-\infty}^{\infty}\mathrm{E}X_{0,m}(t)X_j(s)\right)dtds\int_{-1}^{1}K^2(u)du$$

and

$$\frac{1}{h}\sum_{\ell=-h}^{h}\sum_{k=-h}^{h}\sum_{i=\max(1,1-\ell)}^{\min(N,N-\ell)}\sum_{j=\max(1,1-g)}^{\min(N,N-g)}K\left(\frac{\ell}{h}\right)K\left(\frac{k}{h}\right)\iint a_{j-i,m}^{(2)}(t,t)a_{j-i+g-\ell}^{(2)}(s,s)dtds$$

$$\to \left(\int\sum_{\ell=-\infty}^{\infty}\mathrm{E}X_0(t)X_{\ell,m}(t)dt\right)^2\int_{-1}^{1}K^2(u)du,$$

completing the proof of (4.14). $\qquad\square$

4.2. Approximations with finite dimensional processes

Based on the result in Section 4.1, we now assume that

$$X_i(t), -\infty < i < \infty \text{ is an } m\text{–dependent stationary sequence,} \qquad (4.43)$$

$$\mathrm{E}X_i(t) = 0 \ \text{ and } \ \mathrm{E}\|X_0\|^4 < \infty. \qquad (4.44)$$

First we replace the bivariate cumulant function $\psi_{\ell,r,p}$ of Section 4.1 with the four variate version

$$\chi_{\ell,r,p}(t,s,t',s') = \mathrm{E}[X_0(t)X_\ell(s)X_r(t')X_p(s')] - a_\ell(t,s)a_{p-r}(t',s') - a_r(t,t')a_{p-\ell}(s,s')$$
$$- a_p(t,s')a_{r-\ell}(s,t').$$

Lemma 4.4. *If* (4.43) *and* (4.44) *are satisfied, then we have*

$$\left\| \sum_{\ell,r,p=-\infty}^{\infty} \chi_{\ell,r,p}(t,s,t',s') \right\| < \infty.$$

Proof. Using stationarity arguments, we need to prove only that

$$\left\| \sum_{\ell,r,p=0}^{\infty} \chi_{\ell,r,p}(t,s,t',s') \right\| < \infty.$$

Let $D = \{(\ell,r,p) : \ell,r,p \geq 0\}$ and $D_1 = \{(\ell,r,p) : 0 \leq \ell \leq r \leq p\}$. If $(\ell,r,p) \in D_1$ and $p - r > m$, then $\chi_{\ell,r,p} = 0$ since each term in the definition of $\chi_{\ell,r,p}$ equals 0 in this case due to the m–dependence. Similarly, if $\ell > m$ or $r - \ell > m$, then $\chi_{\ell,r,p}$ equals 0 for all $(\ell,r,p) \in D_1$. Therefore $\{(\ell,r,p) \in D_1 : \chi_{\ell,r,p} \neq 0\} \subseteq \{(\ell,r,p) : 0 \leq \ell \leq m, 0 \leq r \leq 2m, p - r \leq m\}$ and the last set has no more than $6(m+1)^3$ elements. Hence

$$\left\| \sum_{D_1} \chi_{\ell,r,p}(t,s,t',s') \right\| < \infty \tag{4.45}$$

since only finitely many terms are different from zero in the sum. The other subsets of D can be handled similarly so the details are omitted. $\qquad\square$

Let

$$L_N(t,s,t',s') = \frac{N}{h} \sum_{\ell,g=-h}^{h} K\left(\frac{\ell}{h}\right) K\left(\frac{g}{h}\right) \mathrm{var}(\tilde{\gamma}_\ell(t,s),\tilde{\gamma}_g(t',s')).$$

Lemma 4.5. *If* (4.43) *and* (4.44) *are satisfied, then we have that*

$$\|L_N - L\| \to 0.$$

where L is defined in (2.10).

Proof. Following the proof of Lemma 4.3 we write

$$L_N(t,s,t',s') = q_{1,N}(t,s,t',s') + q_{2,N}(t,s,t',s') + q_{3,N}(t,s,t',s'),$$

where

$$q_{1,N}(t,s,t',s') = \frac{1}{h} \sum_{g,\ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K\left(\frac{\ell}{h}\right) K\left(\frac{g}{h}\right) \bar{\varphi}_N(r,\ell,g) \chi_{\ell,r,r+g}(t,s,t',s'),$$

$$q_{2,N}(t,s,t',s') = \frac{1}{h} \sum_{g,\ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K\left(\frac{\ell}{h}\right) K\left(\frac{g}{h}\right) \bar{\varphi}_N(r,\ell,g) a_{r+g}(t',s') a_{r-\ell}(t',s),$$

$$q_{3,N}(t,s,t',s') = \frac{1}{h} \sum_{g,\ell=-h}^{h} \sum_{r=-(N-1)}^{N-1} K\left(\frac{\ell}{h}\right) K\left(\frac{g}{h}\right) \bar{\varphi}_N(r,\ell,g) a_r(t,t') a_{r+g-\ell}(s,s'),$$

23

and $\bar{\varphi}_N(r, \ell, g)$ is defined an (4.31). If we write

$$L(t, s, t', s') = L^{(1)}(t, s, t', s') + L^{(2)}(t, s, t', s'),$$

where

$$L^{(1)}(t, s, t', s') = C(t, s)C(t', s') \int_{-c}^{c} K^2(z)dz$$

and

$$L^{(2)}(t, s, t', s') = C(t, t')C(s, s') \int_{-c}^{c} K^2(z)dz,$$

then by the triangle inequality we get

$$\|L_N - L\| \le \|q_{1,N}\| + \|q_{2,N} - L^{(1)}\| + \|q_{3,N} - L^{(2)}\|.$$

Clearly,

$$\|q_{1,N}\| \le \frac{1}{h} \sup_{|x| \le c} K^2(x) \left\| \sum_{\ell, r, q = -\infty}^{\infty} \chi_{\ell, r, q} \right\| \to 0, \quad \text{as } N \to \infty,$$

on account of Lemma 4.4. By assumption (2.8) for all large enough N we have that $h + N - 1 \ge m$. Since $a_j = 0$ for all $|j| > m$, by a change of variables we have for all large enough N

$$q_{2,N}(t, s, t', s') = \frac{1}{h} \sum_{|u|, |v| < h+N} \sum_{r=b_1}^{b_2} K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) \bar{\varphi}_N(r, r-v, u-r) a_u(t, s') a_v(t', s)$$

$$= \frac{1}{h} \sum_{|u|, |v| \le m} \sum_{r=b_1}^{b_2} K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) \bar{\varphi}_N(r, r-v, u-r) a_u(t, s') a_v(t', s),$$

where $b_1 = b_1(u, v, N)$ and $b_2 = b_2(u, v, N)$ are defined in (4.32) and (4.33), respectively. Define

$$q_{2,N,1}(t, s, t', s') = \frac{1}{h} \sum_{|u|, |v| \le m} \sum_{r=b_1}^{b_2} K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) a_u(t, s') a_v(t', s),$$

$$q_{2,N,2}(t, s, t', s') = \frac{1}{h} \sum_{|u|, |v| \le m} \sum_{r=b_1}^{b_2} K^2\left(\frac{r}{h}\right) a_u(t, s') a_v(t', s),$$

$$q_{2,N,3}(t, s, t', s') = \frac{1}{h} \sum_{|u|, |v| \le m} \sum_{r=-h}^{h} K^2\left(\frac{r}{h}\right) a_u(t, s') a_v(t', s).$$

Since $|\bar{\varphi}_N(r, r-v, u-r) - 1| \le 2(3m+h)/N$ for all $1 \le r \le N$ and $|u|, |v| \le m$ we conclude by Fubini's theorem that as $N \to \infty$,

$$\|q_{2,N} - q_{2,N,1}\| \le \frac{2Q(3m+h)}{N} \sup_{|x| \le 1} K^2(x) \to 0,$$

24

where

$$Q = \left(\iint \left(\sum_{|u| \leq m} |a_u(t, s')| \right)^2 dt ds' \iint \left(\sum_{|v| \leq m} |a_v(t', s)| \right)^2 dt' ds \right)^{1/2}.$$

Using (2.7) one can find a constant η such that

$$\sup_{|u| \leq m} \left| K\left(\frac{u-r}{h}\right) - K\left(\frac{r}{h}\right) \right| \leq \eta \frac{m}{h} \tag{4.46}$$

and therefore

$$\sup_{|u|,|v| \leq m} \left| K\left(\frac{u-r}{h}\right) K\left(\frac{v-r}{h}\right) - K^2\left(\frac{r}{h}\right) \right| \leq 2\eta \frac{m}{h} \sup_{|x| \leq 1} |K(x)|. \tag{4.47}$$

By (4.46) and (4.47) we obtain that

$$\|q_{2,N,1} - q_{2,N,2}\| \leq 2\eta Q \frac{m}{h} \sup_{|x| \leq 1} |K(x)| \;\; \rightarrow \;\; 0, \quad \text{as} \quad N \to \infty.$$

It follows from the definitions of $b_1 = b_1(u, v, N)$ and $b_2 = b_2(u, v, N)$ in (4.32) and (4.33) that

$$\|q_{2,N,2} - q_{2,N,3}\| \leq \frac{2Qm}{h} \sup_{|x| \leq 1} K^2(x) \;\; \rightarrow \;\; 0, \quad \text{as} \quad N \to \infty.$$

Finally,

$$\|q_{2,N,3} - L^{(1)}\| \leq Q \left| \frac{1}{h} \sum_{r=-h}^{h} K^2\left(\frac{r}{h}\right) - \int_{-1}^{1} K^2(z) dz \right| \;\; \rightarrow \;\; 0,$$

as $N \to \infty$, since K is Riemann integrable. This also concludes the proof of

$$\|q_{2,N} - L^{(1)}\| \;\; \rightarrow \;\; 0, \quad \text{as} \quad N \to \infty.$$

Similar arguments yield

$$\|q_{3,N} - L^{(2)}\| \;\; \rightarrow \;\; 0, \quad \text{as} \quad N \to \infty,$$

completing the proof of Lemma 4.5. $\qquad\qquad\square$

Let $\phi_i(t), 1 \leq i < \infty$ be an orthonormal basis of $L^2([0,1], \mathbb{R})$. By the Karhunen–Loéve expansion we can write

$$X_i(t) = \sum_{\ell=1}^{\infty} \langle X_i, \phi_\ell \rangle \phi_\ell(t).$$

Define

$$X_i^{(d)}(t) = \sum_{\ell=1}^{d} \langle X_i, \phi_\ell \rangle \phi_\ell(t)$$

25

and correspondingly $\bar{Z}_N^{(d)} = \bar{C}_N^{(d)}(t,s) - \mathrm{E}\bar{C}_N^{(d)}(t,s)$, where

$$\bar{C}_N^{(d)}(t,s) = \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \bar{\gamma}_i^{(d)}(t,s)$$

with

$$\bar{\gamma}_i^{(d)}(t,s) = \tilde{\gamma}_{i,N}(t,s) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{j=1}^{N-i} X_j^{(d)}(t) X_{j+i}^{(d)}(s), & i \geq 0 \\[3mm] \dfrac{1}{N} \displaystyle\sum_{j=1-i}^{N} X_j^{(d)}(t) X_{j+i}^{(d)}(s), & i < 0. \end{cases}$$

It follows from the Karhunen–Loéve theorem that

$$\mathrm{E}\|X_0 - X_0^{(d)}\|^2 \to 0, \quad \text{as } d \to \infty. \tag{4.48}$$

Let

$$C^{(d)}(t,s) = \sum_{\ell=-m}^{m} \mathrm{E}X_0^{(d)}(t)\mathrm{E}X_\ell^{(d)}(s).$$

Lemma 4.6. *If* (4.43) *and* (4.44) *are satisfied, then we have that*

$$\lim_{d\to\infty} \limsup_{N\to\infty} \frac{N}{h}\mathrm{E}\|\tilde{Z}_N - \bar{Z}_N^{(d)}\|^2 = 0. \tag{4.49}$$

Also, for each $d \geq 1$,

$$\lim_{N\to\infty} \frac{N}{h} \iint \mathrm{var}(\bar{Z}_N^{(d)}(t,s)) dtds = \left(\|C^{(d)}\|^2 + \left(\int C^{(d)}(t,t)dt\right)^2\right)\int_{-c}^{c} K^2(u)du, \tag{4.50}$$

and

$$\lim_{N\to\infty} \frac{N}{h} \iint \mathrm{cov}(\tilde{Z}_N(t,s), \tilde{Z}_N^{(d)}(t,s)) dtds \tag{4.51}$$
$$= \left\{\iint \left(\sum_{\ell=-m}^{m} \mathrm{E}X_0(t)X_\ell^{(d)}(s)\right)\left(\sum_{j=-m}^{m} \mathrm{E}X_0^{(d)}(t)X_\ell(s)\right) dtds\right.$$
$$\left. + \left(\int \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)X_\ell^{(d)}(t)dt\right)^2\right\}\int_{-c}^{c} K^2(u)du.$$

Proof. Using the Cauchy–Schwarz inequality and stationarity we have

$$\iint \left|\mathrm{E}[(X_0(t) - X_0^{(d)}(t))X_i(s)]\mathrm{E}X_0(t)X_j(s)\right| dtds \tag{4.52}$$
$$\leq \left\{\iint [\mathrm{E}(X_0(t) - X_0^{(d)}(t))X_i(s)]^2 dtds\right\}^{1/2} \left\{\iint [\mathrm{E}X_0(t)\mathrm{E}X_j(s)]^2 dtds\right\}^{1/2}$$

$$\leq \left\{ \iint \mathrm{E}(X_0(t) - X_0^{(d)}(t))^2 \mathrm{E}X_i^2(s) dt ds \right\}^{1/2} \left\{ \iint \mathrm{E}X_0^2(t) \mathrm{E}X_j^2(s) dt ds \right\}^{1/2}$$

$$= (\mathrm{E}\|X_0 - X_0^{(d)}\|^2)^{1/2} (\mathrm{E}\|X_0\|^2)^{3/2}$$

and similarly

$$\iint \left| \mathrm{E}[X_0^{(d)}(t)(X_i(s) - X_i^{(d)}(s))]\mathrm{E}X_0^{(d)}(t)X_j^{(d)}(s) \right| dt ds \tag{4.53}$$

$$\leq (\mathrm{E}\|X_0 - X_0^{(d)}\|^2)^{1/2}(\mathrm{E}\|X_0^{(d)}\|^2)^{3/2}$$

$$\leq (\mathrm{E}\|X_0 - X_0^{(d)}\|^2)^{1/2}(\mathrm{E}\|X_0\|^2)^{3/2}.$$

Hence by elementary calculations we conclude from these inequalities

$$\iint \left| \mathrm{E}X_0(t)X_i(s)\mathrm{E}X_0(t)X_j(s) - \mathrm{E}X_0^{(d)}(t)X_i^{(d)}(s)\mathrm{E}X_0^{(d)}(t)X_j^{(d)}(s) \right| dt ds$$

$$\leq A(\mathrm{E}\|X_0 - X_0^{(d)}\|^2)^{1/2}(\mathrm{E}\|X_0\|^2)^{3/2}.$$

with some constant A. Thus we get as $d \to \infty$ that

$$\left\| \sum_{\ell=-m}^{m} \mathrm{E}X_0^{(d)}(t)\mathrm{E}X_\ell^{(d)}(s) \right\| \rightarrow \left\| \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)\mathrm{E}X_\ell(s) \right\|, \tag{4.54}$$

$$\int \sum_{\ell=-m}^{m} \mathrm{E}X_0^{(d)}(t)\mathrm{E}X_\ell^{(d)}(t) dt \rightarrow \int \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)\mathrm{E}X_\ell(t) dt \tag{4.55}$$

$$\iint \left(\sum_{\ell=-m}^{m} \mathrm{E}X_0(t)X_\ell^{(d)}(s) \right) \left(\sum_{j=-m}^{m} \mathrm{E}X_0^{(d)}(t)X_\ell(s) \right) dt ds \tag{4.56}$$

$$\rightarrow \left\| \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)\mathrm{E}X_\ell(s) \right\|,$$

and

$$\int \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)X_\ell^{(d)}(t) dt \rightarrow \int \sum_{\ell=-m}^{m} \mathrm{E}X_0(t)\mathrm{E}X_\ell(t) dt. \tag{4.57}$$

On account of (4.48), the result in (4.49) follows from (4.12), (4.50), (4.51) and (4.54)–(4.57).

Lemma 4.5 implies (4.50). The proof of (4.51) goes along the lines of (4.14) but it is much simpler since (4.15) always satisfied for m–dependent random functions. Hence the details are omitted. □

27

4.3. Normal approximation in case of finite dimensional m–dependent processes

Based on the result in Sections 4.1 and 4.2 we can and will assume in this section that

$$X_i(t), -\infty < i < \infty \text{ is an } m\text{–dependent } d\text{–dimensional stationary sequence}, \quad (4.58)$$

i.e.

$$X_i(t) = \sum_{\ell=1}^{d} \langle X_i, \phi_\ell \rangle \phi_\ell(t), \quad (4.59)$$

where $\phi_\ell(t), \ell \geq 1$ is a basis of L^2. Let

$$C_N^*(t, s) = \sum_{i=-\infty}^{\infty} K\left(\frac{i}{h}\right) \gamma_i^*(t, s)$$

with for all $-N < i < N$

$$\gamma_i^*(t, s) = \frac{1}{N} \sum_{j=1}^{N} X_j(t) X_{j+i}(s).$$

First we show that the difference between $Z_N^*(t, s) = C_N^*(t, s) - \mathrm{E}C_N^*(t, s)$ and $\tilde{Z}_N(t, s)$ is small.

Lemma 4.7. *If* (4.58),(2.5)–(2.8) *are satisfied, then we have*

$$\frac{N}{h} \|Z_N^* - \tilde{Z}_N\|^2 = o_P(1), \quad as \quad N \to \infty.$$

Proof. Let

$$s_{\ell, N} = \begin{cases} \{j : \ N - \ell < j \leq N\}, & \text{if } \ell \geq 0 \\ \{j : \ 1 \leq j \leq 1 - \ell\}, & \text{if } \ell < 0. \end{cases}$$

Then according to the definitions of $\tilde{Z}_N$ and Z_N^* we have

$$\mathrm{E}\|\tilde{Z}_N - Z_N^*\|^2 = \frac{1}{N^2} \iint \sum_{\ell=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} K\left(\frac{\ell}{h}\right) K\left(\frac{p}{h}\right) \quad (4.60)$$

$$\times \sum_{j \in s_{\ell, N}} \sum_{i \in s_{p, N}} \left(\mathrm{E}X_j(t)X_{j+\ell}(s)X_i(t)X_{i+p}(s) - a_\ell(t, s)a_p(t, s) \right) dt ds.$$

Using the m–dependence of the X_i's, one can verify along the lines of the arguments used in Lemma 4.3 that the right side of (4.60) is $O(h^2)$. Thus the result follows from (2.8) via Markov's inequality. $\qquad\square$

Using (4.59) we have

$$Z_N^*(t, s) = \sum_{\ell=-h}^{h} K\left(\frac{\ell}{h}\right) (\gamma_\ell^*(t, s) - \mathrm{E}\gamma_\ell^*(t, s))$$

28

$$= \sum_{r=1}^{d} \sum_{p=1}^{d} \left\{ \frac{1}{N} \sum_{\ell=-h}^{h} \sum_{j=1}^{N} (\xi_{r,j}\xi_{p,j+\ell} - \mathrm{E}\xi_{r,j}\xi_{p,j+\ell}) K(\ell/h) \right\} \phi_r(t)\phi_p(s),$$

where

$$\xi_{r,j} = \langle X_j, \phi_r \rangle.$$

In order to show that $(N/h)^{1/2} Z_N^*(t,s)$ can be approximated with a Gaussian process, we begin by establishing that the d^2–dimensional vector

$$\left\{ \frac{1}{(Nh)^{1/2}} \sum_{\ell=-h}^{h} \sum_{j=1}^{N} (\xi_{r,j}\xi_{p,j+\ell} - \mathrm{E}\xi_{r,j}\xi_{p,j+\ell}) K(\ell/h), 1 \le r,p \le d \right\} \xrightarrow{\mathcal{D}} \mathcal{N}_{d^2}, \qquad (4.61)$$

where $\mathcal{N}_{d^2}$ is a d^2–dimensional normal random vector. By the Cramér–Wold device it is sufficient to show that

$$\sum_{r=1}^{d} \sum_{p=1}^{d} \beta_{r,p} \frac{1}{(Nh)^{1/2}} \sum_{\ell=-h}^{h} \sum_{j=1}^{N} (\xi_{r,j}\xi_{p,j+\ell} - \mathrm{E}\xi_{r,j}\xi_{p,j+\ell}) K(\ell/h) \xrightarrow{\mathcal{D}} \mathcal{N}, \qquad (4.62)$$

for any constants $\beta_{r,p}, 1 \le r,p \le d$, where $\mathcal{N}$ denotes a normal random variable. By the definition of $\xi_{i,j}$,

$$\xi_{r,j}\xi_{p,j+\ell} = \iint X_j(t) X_{j+\ell}(s) \phi_r(t) \phi_p(s) dt ds,$$

and therefore

$$\sum_{r=1}^{d} \sum_{p=1}^{d} \beta_{r,p} \frac{1}{(Nh)^{1/2}} \sum_{\ell=-h}^{h} \sum_{j=1}^{N} (\xi_{r,j}\xi_{p,j+\ell} - \mathrm{E}\xi_{r,j}\xi_{p,j+\ell}) K(\ell/h)$$

$$= \frac{1}{(Nh)^{1/2}} \sum_{\ell=-h}^{h} K\left(\frac{\ell}{h}\right) \sum_{j=1}^{N} \iint \left\{ (X_j(t) X_{j+\ell}(s) \right.$$

$$\left. - \mathrm{E}X_j(t) X_{j+\ell}(s)) \sum_{r=1}^{d} \sum_{p=1}^{d} \beta_{r,p}\phi_r(t)\phi_p(s) \right\} dt ds.$$

Therefore (4.62) follows if we prove that for any $f \in L^2([0,1]^2, \mathbb{R})$

$$\frac{1}{(Nh)^{1/2}} \sum_{\ell=-h}^{h} K\left(\frac{\ell}{h}\right) \sum_{j=1}^{N} \alpha_{j,\ell} \xrightarrow{\mathcal{D}} \mathcal{N}, \qquad (4.63)$$

where

$$\alpha_{j,\ell} = \iint (X_j(t) X_{j+\ell}(s) - \mathrm{E}X_j(t) X_{j+\ell}(s)) f(t,s) dt ds.$$

The proof of (4.63) is based on a blocking argument. We write

$$\Delta_N = \sum_{\ell=-h}^{h} \sum_{j=1}^{N} \alpha_{j,\ell} K\left(\frac{\ell}{h}\right) = \sum_{i=1}^{Q} R_i + \sum_{i=1}^{Q} D_i + T',$$

29

where

$$R_i = \sum_{j \in B_i} \sum_{\ell=-h}^{h} \alpha_{j,\ell} K\left(\frac{\ell}{h}\right), \quad D_i = \sum_{j \in b_i} \sum_{\ell=-h}^{h} \alpha_{j,\ell} K\left(\frac{\ell}{h}\right) \quad \text{and} \quad T' = \sum_{j \in T} \sum_{\ell=-h}^{h} \alpha_{j,\ell} K\left(\frac{\ell}{h}\right)$$

with

$$B_i = \{j : 1 + (i-1)M + 2(i-1)h \le j < 1 + iM + 2(i-1)h\}, \quad Q = \lfloor N/(2h + M) \rfloor$$
$$b_i = \{j : 1 + iM + 2(i-1)h \le j < 1 + i(M + 2h)\}, \quad T = \{j : Q(2h + M) \le j \le N\}$$

and $M > h$ is a numerical sequence. It follows from assumption (4.58) that $R_1, R_2, \ldots, R_Q$ are independent and identically distributed random variables with zero mean. Similarly, $D_1, D_2, \ldots, D_Q$ are independent and identically distributed random variables with zero mean.

The following elementary lemma will be useful to get sharp upper bounds for the moments of the blocks.

Lemma 4.8. *Suppose $\{Y_i\}_{i=1}^{\infty}$ is an m–dependent sequence of random variables. Then for all $\ell \ge 0$, the sequence of random vectors $\{(Y_i, Y_{i+\ell})\}_{i=1}^{\infty}$ can be organized into at most $3(m+1)^2$ collections each containing independent random vectors.*

Proof. The sequence $\{(Y_i, Y_{i+\ell})\}_{i=1}^{\infty}$ is $m+\ell$–dependent, it can be organized into $m+\ell+1$ subsets, each containing independent random variables using standard arguments (see, for example, Lemma 2.4 of Berkes et al. (2012)). Hence the result is proven for $\ell \le 2m+1$. From now on we assume that $\ell > 2m + 1$. Let $j^* = \max\{j : j < (\ell - m)/(m + 1)\}$, $k^* = \min\{k : k > (m+\ell)/((m+1)j^*)\}$, and $v^* = ((k^*+1)j^*+1)(m+1)$. Define the set

$$G_{i,k}(p) = \left\{ (Y_{pv^*+(kj^*+j)(m+1)+i}, Y_{pv^*+(kj^*+j)(m+1)+i+\ell}), 0 \le j \le j^* \right\}$$

for $1 \le i \le m + 1$, $0 \le k \le k^*$ and $0 \le p < \infty$. Consider two arbitrary elements of $G_{i,k}(p)$, $\mathbf{X}_r = (Y_{pv^*+(kj^*+r)(m+1)+i}, Y_{pv^*+(kj^*+r)(m+1)+i+\ell})$ and $\mathbf{X}_t = (Y_{pv^*+(kj^*+t)(m+1)+i}, Y_{pv^*+(kj^*+t)(m+1)+i+\ell})$, where, without loss of generality, $0 \le r < t \le j^*$. Clearly, $Y_{pv^*+(kj^*+r)(m+1)+i}$ is independent of $Y_{pv^*+(kj^*+t)(m+1)+i}$ and $Y_{pv^*+(kj^*+t)(m+1)+i+\ell}$, since $r < t$. Also, $Y_{pv^*+(kj^*+r)(m+1)+i+\ell})$ is independent of $Y_{pv^*+(kj^*+t)(m+1)+i+\ell})$ due to $r < t$. Using the definition of j^*, we have

$$|pv^* + (kj^* + r)(m+1) + i + \ell - (pv^* + (kj^* + t)(m+1) + i)| = |\ell - (r-t)(m+1)|$$
$$> \ell - j^*(m+1) > m.$$

Hence $Y_{pv^*+(kj^*+r)(m+1)+i+\ell}$ and $Y_{pv^*+(kj^*+t)(m+1)+i}$ are independent, establishing the independence of $\mathbf{X}_r$ and $\mathbf{X}_t$. It follows along these lines that the vectors in $G_{i,k}(p)$ are mutually independent. Due to the definition of j^*, $G_{i,k}(p)$ is comprised of $j^* + 1$ independent random variables. Further, according to the definition of v^*, $G_{i,k}(p)$ and $G_{i,k}(p')$ are independent for all integers $p \ne p'$. By the definitions of k^* and j^* we have that

$$k^* \le \frac{m+\ell}{(m+1)j^*} \le \frac{m+\ell}{\ell - 2m - 1} \le 3m + 2, \quad \text{where we used} \quad j^* \ge \frac{\ell - m}{m+1} - 1.$$

30

It follows that the union of the at most $(3m + 3)(m + 1)$ sets $\cup_{p=0}^{\infty} G_{i,k}(p), 1 \leq i \leq m + 1, 0 \leq k \leq k^*$ contains $\{(Y_i, Y_{i+\ell})\}_{i=1}^{\infty}$.

$\square$

Lemma 4.9. *If (4.58),(2.5)–(2.7) are satisfied, $h = h(N) \to \infty$ and $M/h \to \infty$, then we have*

$$\mathrm{E}\left((hM)^{-1/2} R_1\right)^2 \to \int \cdots \int L(t, s, t', s') f(t, s) f(t', s') dt ds dt' ds' \tag{4.64}$$

where L is defined in (2.10),

$$\mathrm{E}|R_1|^{2+\delta/2} = O(h^{2+\delta/2} M^{1+\delta/4}) \tag{4.65}$$

and

$$\mathrm{E}D_1^2 = O(h^2). \tag{4.66}$$

Proof. The assertions in (4.64) and (4.66) can be established along the lines of the proof Lemma 4.5. Due to the m–dependence assumed in (4.58), the proofs are much simpler in the present case.
By Petrov (1995, p. 58)

$$\mathrm{E}|R_1|^{2+\delta/2} \leq (2h + 1)^{1+\delta/2} \sum_{\ell=-h}^{h} K^{2+\delta/2}\left(\frac{\ell}{h}\right) \mathrm{E}\left(\sum_{i=1}^{M+1} \alpha_{i,\ell}\right)^{2+\delta/2}.$$

Using Lemma 4.8 we can write $\sum_{i=1}^{M+1} \alpha_{i,\ell}$ as the sum of no more than $3(m + 1)^2$ sums, each sum is based on i.i.d. random variables. Hence via the triangle and Rosenthal's inequalities we get

$$\left(\mathrm{E}\left(\sum_{i=1}^{M+1} \alpha_{i,\ell}\right)^{2+\delta/2}\right)^{1/(2+\delta/2)} \leq c_0 M^{1/2}$$

with some constant c_0, completing the proof of (4.65).

$\square$

Lemma 4.10. *If (4.58) and (2.5)–(2.9) are satisfied, then we have $(Nh)^{-1/2}\Delta_N$ converges in distribution to a normal random variable with zero mean and variance $\int \cdots \int L(t, s, t', s') f(t, s) f(t', s') dt ds dt' ds'$.*

Proof. Under assumption (2.9) one can find a sequence M such that $M/h \to \infty$ and $h(M/N)^{\delta/(4+\delta)} \to 0$ and therefore using (4.66) of Lemma 4.9 and the independence of the D_i's we obtain that

$$\mathrm{E}\left(\frac{1}{\sqrt{Nh}}\sum_{i=1}^{Q} D_i\right)^2 = O\left(\frac{1}{Nh}Qh^2\right) = O\left(\frac{h}{M}\right) = o(1)$$

and therefore

$$\frac{1}{\sqrt{Nh}}\sum_{i=1}^{Q} D_i \xrightarrow{P} 0.$$

31

Similar argument yields

$$\frac{1}{\sqrt{Nh}}T' \xrightarrow{P} 0.$$

Using now (4.64) and (4.65) we conclude

$$\frac{\left(\sum_{i=1}^{Q} \mathrm{E}|R_i|^{2+\delta/2}\right)^{1/(2+\delta/2)}}{\left(\sum_{i=1}^{Q} \mathrm{E}R_i^2\right)^{1/2}} = O(1)\frac{(Qh^{2+\delta/2}M^{1+\delta/4})^{1/(2+\delta/2)}}{(hQM)^{1/2}}$$

$$= O(1)\frac{N^{1/(2+\delta/2)}hM^{\delta/(8+2\delta)}}{(Nh)^{1/2}} \to 0.$$

Now Lyapunov's theorem; see Petrov (1995, p. 126) and (4.64) imply Lemma 4.10. □

Lemma 4.11. *If* (4.58),(2.5)–(2.9) *and are satisfied, then we can define Gaussian processes* $\Gamma_N(t,s)$ *with* $\mathrm{E}\Gamma_N(t,s) = 0$, $\mathrm{E}\Gamma_N(t,s)\Gamma_N(t',s') = L(t,s,t',s')$ *such that*

$$\|(N/h)^{1/2}Z_N^* - \Gamma_N\| = o_P(1), \quad as \ N \to \infty.$$

Proof. As we argued at the beginning of this section, Lemma 4.10 yields that (4.61) holds with $\mathcal{N}_{d^2} = \{\mathcal{N}_{d^2}(r,p), 1 \le r, p \le d\}$, $\mathrm{E}\mathcal{N}_{d^2}(r,p) = 0$ and

$$\mathrm{E}\mathcal{N}_{d^2}(r,p)\mathcal{N}_{d^2}(r',p') = \int \cdots \int L(t,s,t',s')\phi_r(t)\phi_p(s)\phi_{r'}(t')\phi_{p'}(s')dtdsdt'ds'.$$

By the Skorokhod–Dudley–Wichura representation, see Shorack & Wellner (1986), p. 47, we can define $\mathcal{N}_{d^2}^{(N)} = \{\mathcal{N}_{d^2}^{(N)}(r,p), 1 \le r, p \le d\}$, a copy of $\mathcal{N}_{d^2}$ such that

$$\max_{1 \le r,p \le d}\left|(Nh)^{-1/2}\sum_{\ell=-h}^{h}\sum_{j=1}^{N}(\xi_{r,j}\xi_{p,j+\ell} - \mathrm{E}\xi_{r,j}\xi_{p,j+\ell})K(\ell/h) - \mathcal{N}_{d^2}^{(N)}(r,p)\right| = o_P(1). \quad (4.67)$$

Clearly,

$$\Gamma_N(t,s) = \sum_{r=1}^{d}\sum_{p=1}^{d}\mathcal{N}_{d^2}^{(N)}(r,p)\phi_r(t)\phi_p(s)$$

is a Gaussian process with mean zero and $\mathrm{E}\Gamma_N(t,s)\Gamma_N(t',s') = L(t,s,t',s')$. Using now (4.67), Lemma 4.11 follows. □

Next we show if (2.12) is satisfied then the conclusion of Lemma 4.11 holds assuming only (2.8) instead of the much stronger restriction (2.9) on h.

Lemma 4.12. *If* (4.58),(2.5)–(2.8) *and* (2.12) *are satisfied, then we can define Gaussian processes* $\Gamma_N(t,s)$ *with* $\mathrm{E}\Gamma_N(t,s) = 0$, $\mathrm{E}\Gamma_N(t,s)\Gamma_N(t',s') = L(t,s,t',s')$ *such that*

$$\|(N/h)^{1/2}Z_N^* - \Gamma_N\| = o_P(1), \quad as \ N \to \infty.$$

Proof. Following the proof of Lemma (4.5) we can verify that

$$\mathrm{E}R_1^4 = O(h^2 M^2). \tag{4.68}$$

In the proof of Lemma 4.10 when Lyapunov's condition is verified we now use (4.68) instead of (4.65). Hence Lemma 4.10 holds assuming only (2.8) when (2.12) holds. Now Lemma 4.12 follows from the central limit theorem of Lemma 4.10 using the Skorokhod–Dudley–Wichura representation theorem, see Shorack & Wellner (1986), p. 47, as in Lemma 4.11. $\qquad\square$

4.4. Proofs of Theorems 2.1–2.3

Proof of Theorem 2.1. Using the results in Sections 4.1 and 4.2 with Lemma 4.11, for every $\varepsilon > 0$ there are integers m_0 and d_0 such that

$$\limsup_{N\to\infty} \Pr\left\{\|(N/h)^{1/2}Z_N - \Gamma_{N,d,m}\| > \varepsilon\right\} < \varepsilon \quad \text{for all} \quad m > m_0, d > d_0,$$

where

$$\{\Gamma_{N,d,m}(t,s), 0 \le t,s \le 1\} \stackrel{\mathcal{D}}{=} \left\{\sum_{r=1}^{d}\sum_{p=1}^{d}\mathcal{N}_{d^2}^{(m)}(r,p)\phi_r(t)\phi_p(s), 0 \le t,s \le 1\right\},$$

$$\Gamma_{d,m}(t,s) = \sum_{r=1}^{d}\sum_{p=1}^{d}\mathcal{N}_{d^2}^{(m)}(r,p)\phi_t(t)\phi_p(s),$$

$\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis of L^2, and $\{\mathcal{N}_{d^2}^{(m)}(r,p), 1 \le r,p \le d\}$ is d^2–dimensional normal with zero mean and

$$\mathrm{E}\mathcal{N}_{d^2}^{(m)}(r,p)\mathcal{N}_{d^2}^{(m)}(r',p') = \int\!\!\cdots\!\!\int L^{(m)}(t,s,t',s')\phi_r(t)\phi_p(s)\phi_{r'}(t')\phi_{p'}(s')dtdsdt'ds'$$

with

$$L^{(m)}(t,s,t',s') = [C_m(t,s)C_m(t',s') + C_m(t,t')C_m(s,s')]\int_{-1}^{1}K^2(z)dz,$$

$$C_m(t,s) = \sum_{\ell=-m}^{m}\mathrm{cov}(X_{0,m}(t), X_{\ell,m}(s)),$$

and the variables $X_{i,m}$ are defined in (2.4). Using (4.7) we conclude $\|L^{(m)} - L\| \to 0$, as $m \to \infty$ and therefore

$$\{\mathcal{N}_{d^2}^{(m)}(r,p), 1 \le r,p \le d\} \stackrel{\mathcal{D}}{\to} \{\mathcal{N}(r,p), 1 \le r,p \le d\}, \quad \text{as } m \to \infty,$$

where $\{\mathcal{N}(r,p), 1 \le r,p < \infty\}$ is Gaussian with zero mean and

$$\mathrm{E}\mathcal{N}(r,p)\mathcal{N}(r',p') = \int\!\!\cdots\!\!\int L(t,s,t',s')\phi_r(t)\phi_p(s)\phi_{r'}(t')\phi_{p'}(s')dtdsdt'ds'.$$

Hence we can define Gaussian processes $\Gamma_d^{(m)}(t,s)$ such that $\|\Gamma_{d,m} - \Gamma_d^{(m)}\} = o_P(1)$, $\{\Gamma_d^{(m)}(t,s), 0 \le t, s \le 1\} \overset{\mathcal{D}}{=} \{\Gamma_d(t,s), 0 \le t, s \le 1\}$ and

$$\Gamma_d(t,s) = \sum_{r=1}^{d} \sum_{p=1}^{d} \mathcal{N}(r,p)\phi_t(t)\phi_p(s).$$

Observing that $\|\Gamma_d - \Gamma\| = o_P(1)$ as $d \to \infty$, where $\Gamma(t,s) = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \mathcal{N}(r,p)\phi_t(t)\phi_p(s)$, the proof of Theorem 2.1 is complete. $\qquad\square$

Proof of Theorem 2.2. We can repeat the proof of Theorem 2.1, we only need to replace Lemma 4.11 with Lemma 4.12. $\qquad\square$

Proof of Theorem 2.3. It is easy to see that

$$\mathrm{E}\hat{C}_N(t,s) = \sum_{\ell=-\infty}^{\infty} K\left(\frac{\ell}{h}\right) \gamma_\ell(t,s) - \frac{1}{N} \sum_{\ell=-\infty}^{\infty} K\left(\frac{\ell}{h}\right) \ell\gamma_\ell(t,s)$$

and

$$\left\| \sum_{\ell=-\infty}^{\infty} K\left(\frac{\ell}{h}\right) \ell\gamma_\ell \right\| = O(1).$$

Let $\varepsilon > 0$. Next we write

$$\sum_{\ell=-\infty}^{\infty} K\left(\frac{\ell}{h}\right) \gamma_\ell(t,s) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell(t,s) + f_{N,1,\varepsilon}(t,s) + f_{N,2,\varepsilon}(t,s) - f_{N,3,\varepsilon}(t,s),$$

where

$$f_{N,1,\varepsilon}(t,s) = \sum_{|\ell|\le\varepsilon h} \left(K\left(\frac{\ell}{h}\right) - 1 \right) \gamma_\ell(t,s), \qquad f_{N,2,\varepsilon}(t,s) = \sum_{|\ell|>\varepsilon h} K\left(\frac{\ell}{h}\right) \gamma_\ell(t,s)$$

and

$$f_{N,3,\varepsilon}(t,s) = \sum_{|\ell|>\varepsilon h} \gamma_\ell(t,s).$$

Using assumption (2.14) we conclude by the triangle inequality

$$\|f_{N,2,\varepsilon}(t,s)\| \le \sup_u |K(u)| \sum_{|\ell|>\varepsilon h} \|\gamma_\ell\| \le (h\varepsilon)^{-\mathfrak{q}'} \sup_u |K(u)| \sum_{\ell=-\infty}^{\infty} |\ell|^{\mathfrak{q}'} \|\gamma_\ell\|$$

and similarly

$$\|f_{N,3,\varepsilon}(t,s)\| \le (h\varepsilon)^{-\mathfrak{q}'} \sum_{\ell=-\infty}^{\infty} |\ell|^{\mathfrak{q}'} \|\gamma_\ell\|.$$

By (2.13) we obtain that

$$\lim_{\varepsilon\to 0} \limsup_{h\to\infty} \max_{-\varepsilon h\le\ell\le\varepsilon h} |(K(\ell/h) - 1)/(|\ell|/h)^{\mathfrak{q}} - \mathfrak{K}| = 0$$

and therefore

$$\lim_{\varepsilon\to 0} \limsup_{h\to\infty} \|h^{\mathfrak{q}} f_{N,1,\varepsilon} - \mathfrak{F}\| = 0.$$

Since $\mathfrak{q}' > \mathfrak{q}$ and $h^{\mathfrak{q}}/N \to 0$, the proof of Theorem 2.3 is complete. $\qquad\square$

4.5. Proof of Theorem 3.1

Following the arguments used in the proof of Proposition 1 of Kokoszka & Reimherr (2013) one can show that

$$\max_{1\leq\ell\leq p}\|(N/h)^{1/2}(\hat{s}_\ell\hat{v}_\ell - v_\ell) - \hat{\mathcal{G}}_{\ell,N}\| = o_P(1), \tag{4.69}$$

where

$$\hat{\mathcal{G}}_{\ell,N} = \sum_{k\neq\ell}\frac{v_k(t)}{\lambda_\ell - \lambda_k}\iint Z_N^*(u,s)v_\ell(u)v_k(s)duds$$

and

$$Z_N^*(u,s) = (N/h)^{1/2}(\hat{C}_N(u,s) - C(u,s)).$$

By Theorems 2.1 and 2.3 we have

$$\max_{1\leq\ell\leq p}\|Z_N^* - (\Gamma_N + \mathfrak{a}\mathfrak{F})\| = o_P(1). \tag{4.70}$$

Let $\mathfrak{M}_\ell$ be the mapping from $L^2[0,1]^2 \to L^2[0,1]$ defined by

$$\mathfrak{M}_\ell(f)(t) = \sum_{k\neq\ell}\frac{v_k(t)}{\lambda_\ell - \lambda_k}\iint f(u,s)v_\ell(u)v_k(s)duds.$$

The linear operators $\mathfrak{M}_\ell, 1 \leq \ell \leq p$ are bounded since

$$\|\mathfrak{M}_\ell(f)\|^2 = \sum_{k\neq\ell}(\lambda_\ell - \lambda_k)^{-2}\left(\iint f(u,s)v_\ell(s)v_k(u)dsdu\right)^2 \leq \frac{\|f\|^2}{\alpha_\ell},$$

where

$$\alpha_\ell = \begin{cases} \lambda_1 - \lambda_2, & \text{if } \ell = 1 \\ \min\{\lambda_{\ell-1} - \lambda_\ell, \lambda_\ell - \lambda_{\ell+1}\}, & \text{if } 2 \leq \ell \leq p. \end{cases}$$

The operators $\mathfrak{M}_\ell, 1 \leq \ell \leq p$ are linear and bounded and therefore they are continuous; see Debnath & Mikusinski (1999, p. 27). Hence (4.69) and (4.70) imply that

$$\max_{1\leq\ell\leq p}\|(N/h)^{1/2}(\hat{s}_\ell\hat{v}_\ell - v_\ell) - \mathcal{G}_{\ell,N}\| = o_P(1)$$

where

$$\mathcal{G}_{\ell,N}(t) = \sum_{k\neq\ell}\frac{v_k(t)}{\lambda_\ell - \lambda_k}\iint (\Gamma_N(u,s) + \mathfrak{a}\mathfrak{F}(u,s))v_\ell(u)v_k(s)duds.$$

Similar but somewhat simpler arguments give

$$\max_{1\leq\ell\leq p}|(N/h)^{1/2}\hat{\lambda}_\ell - \lambda_\ell - \mathfrak{g}_{\ell,N}| = o_P(1),$$

where

$$\mathfrak{g}_{\ell,N} = \iint (\Gamma_N(t,s) + \mathfrak{a}\mathfrak{F}(t,s))v_\ell(t)v_\ell(s)dtds.$$

Since $v_\ell(t)v_k(s)$ is a basis in $L^2([0,1]^2, \mathbb{R})$, by the Karhunen–Loéve expansion we have for each N

$$\{\Gamma_N(t,s), 0 \le t, s \le 1\} \overset{\mathcal{D}}{=} \left\{ \sum_{1 \le i,j < \infty} \sigma_{i,j}^{1/2} \mathcal{N}_{i,j} v_i(t) v_j(s), 0 \le t, s \le 1 \right\}, \qquad (4.71)$$

where

$$\sigma_{i,j} = \begin{cases} \lambda_i \lambda_j \displaystyle\int_{-c}^{c} K^2(z) dz, & \text{if } i \ne j, \\[2ex] 2\lambda_i^2 \displaystyle\int_{-c}^{c} K^2(z) dz, & \text{if } i = j. \end{cases}$$

The representation of the limit in Theorem 3.1 follows from (4.71) and the definitions of $\mathfrak{g}_{\ell,N}$ and $\mathcal{G}_{\ell,N}(t)$.

4.6. Acknowledgements

Anderson, T.: *An Introduction to Multivariate Statistical Analysis*, Wiley, New York, 1984.

Andrews, D.: Heteroskedacity and autocorrelation consistent covariance matrix estimation. *Econometrica* **59**(1991), 817–858.

Aue, A., Hörmann, S., Horváth, L., and Reimherr M.: Break detection in the covariance structure of multivariate time series models. *The Annals of Statistics*, **37** (2009), 4046-4087.

Bande, M., Galeano, P. and Manteiga, W.: Functional principle component regression and functional partial least squares regression: an overview and a comparative study. *International Statistical Review* (2015) To appear.

Berkes, I., Horváth, L. and Rice, G.: Weak invariance principles for sums of dependent random functions *Stochastic Processes and their Applications* **123**(2013), 385–403.

Benhenni, K., Hedli–Griche, S., Rachdi, M. and Vieu, P.: Consistency of the regression estimator with functional data under long memory conditions. *Statistics & Probability Letters* **78**(2008), 1043–1049.

Bingham, N.H., Goldie, C.M. and Teugels, J.L.: *Regular Variation.* Cambridge University Press, Cambridge, 1987.

Bosq, D.: *Linear Processes in Function Spaces.* Springer Verlag, 2000.

Brillinger, D.: *Time Series: Data Analysis and Theory.* Classics in Applied Mathematics **36**. SIAM, Philadelphia, PA.

Bühlmann, P.: Locally adaptive lag–window spectral estimation. *Journal of Time Series Analysis.* **17** (1996), 247–270.

Chagny, G. and Roche, A.: Adaptive and minimax estimation of the cumulative distribution function given a functional covariate.*Electronic Journal of Statistics* **8**(2014), 2352–2404.

Cuevas, A.: A partial overview of the theory of statistics with functional data. *Journal of Statistical Planning and Inference* **147**(2014) 1–23.

Dauxois,J., Pousse, A. and Romain, Y.: Asymptotic theory for the principal component analysis of a vector random function: some applications to statistical inference. *Journal of Multivariate Analysis* **12**(1982), 136–154.

Debnath, L. and Mikusinski, P.: *Introduction to Hilbert Spaces with Applications.* Second edition. Academic Press, Inc., San Diego, CA, 1999.

Dehling, H., Sharipov, O., Tewes, J. and Wendler, M.: Sequential block bootstrap in a Hilbert space with application to change point analysis. [arXiv1412.0446] Preprint, 2014.

Dehling, H., Sharipov, O.S. and Wendler, M.: Bootstrap for dependent Hilbert space valued random variables with applications to von Mises statistics. *Journal of Multivariate Analysis* **133**(2015), 200–215.

Ferraty, F. Goia, A. and Vieu, P.: Functional nonparametric model for time series: a fractal approach for dimension reduction. *TEST* **11**(2002), 317–344.

Ferraty, F. and Vieu, P.:*Nonparametric Functional Data Analysis, Theory and Practice*,Springer, New York, 2006.

Hall, P. and Hosseini–Nasab, M.: On properties of functional principal components. *Journal of the Royal Statistical Society: Series B* **68**(2006), 109-126.

Hannan, E. J.: *Multiple Time Series.* Wiley, New York, 1970.

Hörmann, S., Horváth, L. and Reeder, R.: A functional version of the ARCH model. *Econometric Theory* **29**(2013), 138–152.

Hörmann, S., Kidziński, L. and Hallin, M.: Dynamic functional principal components. *Journal of the Royal Statistical Society Ser. B* **77**, (2015), 319-349.

Hörmann, S. and Kokoszka, P.: Weakly dependent functional data. *Annals of Statistics* **38**, (2010) 1845-1884.

Hörmann, S. and Kokoszka, P.: Functional Time Series. *Handbook of Statistics* **30**, (2012) 157186.

Horváth, L., Kokoszka, P. and Reeder, R.: Estimation of the mean of functional time series and a two sample problem. *Journal of the Royal Statistical Society* **75**(2013), 103–122.

Horváth, L. and Rice, G.: Testing for independence between functional time series. *Journal of Econometrics* To appear.

Horváth, L., Rice, G. and Whipple, S.: Adaptive bandwidth selection in the long run covariance estimator of functional time series. *Computational Statistics and Data Analysis* To appear.

Hsing, T. and Eubank, R.: *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators.* Wiley, 2015.

Jirak, M.: On weak invariance principles for sums of dependent random functionals. *Statistics & Probability Letters* **83**(2013), 2291–2296.

Kokoszka, P. and Reimherr, M.: Asymptotic normality of the principal components of functional time series. *Stochastic Processes and their Applications* **123**(2013), 1546-1562.

Kokoszka, P. and Young, G.: KPSS test for functional time series. http://www.stat.colostate.edu/ piotr/kpss.pdf

Ling, N., Ling, L. and Vieu, P.: Nonparametric regression estimation for functional stationary ergodic data with missing at random. *Journal of Statistical Planning and Inference* **162**(2015), 75–87.

Liu, W. and Wu, W.: Asymptotics of spectral density estimates. *Econometric Theory* **26**(2010), 1218-1245.

Mas, A.: Weak convergence for the covariance operators of a Hilbertian linear process. *Stochastic Processes and their Applications* **99**(2002), 117-135.

Mas, A. and Menneteau, L.: Perturbation approach applied to the asymptotic study of random operators. *High dimensional probability III, Progress in Probability* **55**(2003), 127–134.

Masry, E.: Nonparametric regression estimation for dependent functional data: asymptotic normality. *Stochastic Processes and Their Applications* **115**(2005), 155–177.

Panaretos, V., and Tavakoli, S.: Fourier analysis of stationary time series in function space. *Annals of Statistics* **41**(2013), 568-603.

Parzen, E.: On choosing an estimate of the spectral density function of a stationary time series. *Annals of Mathematical Statististics.* **28**(1957), 921-932.

Petrov, V.V.: *Limit Theorems of Probability Theory.* Clarendon Press, Oxford, U.K., 1995.

Pigoli, D., Aston, J.A., Dryden, I.L. and Secchi, P.: Distances and inference for covariance operators. *Biometrika* **101**(2014), 409–422.

Politis, D.M.: Adaptive bandwidth choice. Journal of Nonparametric Statistics **25**(2003), 517–533.

Politis, D.M.: Higher–order accurate, positive semidefinite estimation of large–sample covariance and spectral density matrices. *Econometric Theory* **27**(2011), 703–744.

Politis D. and Romano, J.: Limit theorems for weakly dependent Hilbert space valued random variables with applications to the stationary bootstrap. *Statistica Sinica* **4**(1994), 461–476.

Rachdi, M. and Vieu, P.: Nonparametric regression for functional data: Automatic smoothing parameter selection.*Journal of Statistical Planning and Inference* **137**(2007), 2784–2801.

Ramsay, J.O. and Silverman, B.W.: *Functional Data Analysis.* Springer, 2005.

Rosenblatt, M.: *Stochastic Curves Estimation.* NSF–CBMS Regional Conference Series in Probability and Statistics **3**, Institute of Mathematical Statistics, 1991.

Shao, X. and Wu, W.: Asymptotic spectral theory for nonlinear time series. *Annals of Statistics* **35**(2007), 1773-1801.

Shorack, G.R. and Wellner. J.A.: *Empirical Processes with Applications to Statistics.* Wiley, New York, 1986.

Xiao, H. and Wu, W.: Covariance matrix estimation for stationary time series. *Annals of Statistics* **40**(2012), 466-493.