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Boundary element methods for parabolic boundary control problems

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Abstract

In this paper we analyse constrained optimal Dirichlet boundary control problems subject to the linear heat equation. We propose to use boundary integral equations to solve the coupled optimality system, and we present results on unique solvability and related a priori error estimates for a symmetric Galerkin boundary element method. A numerical example confirms the analytical results.

1 Introduction

Optimal control problems subject to elliptic or parabolic partial differential equations are of great interest, both from a mathematical and an application point of view, see, e.g., [4, 7, 15]. In most cases, the numerical solution by using finite elements is based on the variational formulation of the optimality system. In particular when considering boundary control problems, boundary integral formulations and boundary element methods may be an interesting alternative to finite element methods. In [10] we have introduced and analysed a symmetric Galerkin boundary element method for the solution of Dirichlet boundary control problems subject to the Poisson equation. When describing the solutions of both the primal and adjoint value problems by using boundary integral equations, a variational inequality in the Sobolev trace space $H^{1/2}(\Gamma)$ has to be solved, see also [14]. Since the state enters the adjoint problem as a volume density, an appropriate reformulation of the related Newton potentials by using Bi–Laplace boundary integral operators has to be introduced.

In this paper we analyse the Dirichlet boundary optimal control problem governed by the linear heat equation as a model problem. Several variational formulations for a Dirichlet control in $L_2(\Gamma)$ are considered in [6], and a Galerkin finite element method for the numerical solution of a parabolic Neumann control problem was proposed in [16], where a backward discretization in time was used. Instead, here we propose and analyse the use of boundary element methods to solve the related optimality system, see also [11].

Boundary integral formulations for the heat equation are well established, see, e.g., [2, 3, 9]. In fact, the state u and the adjoint state p can be represented by some layer heat potentials. Since the final state u enters the adjoint heat equation, which is in fact reverse in time, we need to modify the presentation of the related Newton potential by using an auxiliary function which is related to the fundamental solution of the heat equation. With this, similar formulations of boundary integral equations as in the case of stationary boundary control problems [10] are obtained.

This paper is organised as follows. In Sect. 2 we describe the model problem where the Dirichlet control is considered in the boundary energy space $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, where $\Sigma := \Gamma \times (0,T)$ is the boundary of the space time cylinder, and where an equivalent norm is induced by the hypersingular layer heat potential D, see, e.g., [3]. Moreover, we discuss the optimality system, which consists of the primal problem, the adjoint problem, and the optimality condition. In Sect. 3 we first recall the boundary integral equation approach in the case of the heat equation which can be used to describe the primal state u, and the adjoint state p. Since the final state $u(T)$ enters the adjoint equation as a final termination condition, we introduce an auxiliary function to rewrite the related volume potential by using surface potentials only. It turns out that we can state almost the same mapping properties of the resulting boundary integral operators as in the elliptic case [10]. The symmetric Galerkin boundary element approximation of the resulting variational inequality is formulated and analysed in Sect. 4. Finally, in Sect. 5, a numerical example is given which confirms the theorical results.

2 Parabolic Dirichlet boundary control problems

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma = \partial\Omega$ and for a fixed real number $T > 0$, we define the time interval $I := (0, T)$, the space time cylinder $Q := \Omega \times I$, and its boundary $\Sigma := \Gamma \times I$. We consider the model problem to find a Dirichlet control z to minimize the distance of the final temperature $u(\cdot, T)$ from a desired temperature $\overline{u}$, i.e. we consider the cost functional

$$J(u, z) = \frac{1}{2} \int_\Omega [u(x, T) - \overline{u}(x)]^2 \, dx + \frac{\alpha}{2} \langle Dz, z \rangle_\Sigma \tag{2.1}$$

to be minimized subject to the heat equation

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= 0 && \text{for } (x,t) \in Q, \\ u(x,t) &= z(x,t) && \text{for } (x,t) \in \Sigma, \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega, \end{aligned} \tag{2.2}$$

and subject to pointwise control constraints

$$z \in \mathcal{U}_{\mathrm{ad}} := \Big\{ w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \colon z_1(x,t) \le w(x,t) \le z_2(x,t) \text{ for } (x,t) \in \Sigma \Big\}. \tag{2.3}$$

For the definition of the used Sobolev spaces, see, e.g., [1, 3]. We assume $\overline{u}, u_0 \in L_2(\Omega)$, $\alpha \in \mathbb{R}_+$, $z_1, z_2 \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, and $D : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is the hypersingular heat boundary integral operator [3] which defines an equivalent norm in $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. In particular, for $z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ we have

$$(Dz)(x,t) := -\frac{\partial}{\partial n_x} \int_0^t \int_\Gamma \frac{\partial}{\partial n_y} \mathcal{E}(x-y, t-\tau) z(y,\tau)\, ds_y\, d\tau \quad \text{for } (x,t) \in \Sigma,$$

where

$$\mathcal{E}(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} & \text{for } t > 0, \\ 0 & \text{for } t \le 0 \end{cases} \tag{2.4}$$

is the fundamental solution of the heat equation. Let v be a given function defined on $\Omega \times \mathbb{R}_+$ (or $\Gamma \times \mathbb{R}_+$), and let $t_0 \in \mathbb{R}_+$ be arbitrary. Then we define the time reversal map κ_{t_0} by

$$\kappa_{t_0} v(x,t) := v(x, t_0 - t). \tag{2.5}$$

The hypersingular heat boundary integral operator D is $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$–elliptic and self–adjoint with respect to a time–twisted duality, see [3], i.e.,

$$\langle Dz, z \rangle_\Sigma \ge c_1^D \|z\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}, \quad \langle Dz, \kappa_T w \rangle_\Sigma = \langle Dw, \kappa_T z \rangle_\Sigma \quad \text{for all } z, w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma). \tag{2.6}$$

As in [4, 7] we obtain the related optimality conditions as follows:

Theorem 2.1 *Let $(u,z) \in H^{1,\frac{1}{2}}(Q) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ be an optimal solution of the optimal control problem* (2.1)–(2.3). *Then there exists a unique $p \in H^{1,\frac{1}{2}}(Q)$ satisfying the adjoint heat equation*

$$\begin{aligned} -\partial_t p(x,t) - \Delta p(x,t) &= 0 && \textit{for } (x,t) \in Q, \\ p(x,t) &= 0 && \textit{for } (x,t) \in \Sigma, \\ p(x,T) &= u(x,T) - \overline{u}(x) && \textit{for } x \in \Omega, \end{aligned} \tag{2.7}$$

and the optimality condition

$$\langle \alpha \widetilde{D} z - \partial_n p, w - z \rangle_\Sigma \ge 0 \quad \textit{for all } w \in \mathcal{U}_{ad}, \tag{2.8}$$

where

$$\widetilde{D} := \frac{1}{2}(D + \kappa_T D \kappa_T). \tag{2.9}$$

Proof. For a given $z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ there exists a unique solution $u_z \in H^{1,\frac{1}{2}}(Q)$ of the primal heat equation (2.2), see, e.g., [3]. Then the cost functional (2.1) can be rewritten in a reduced form as

$$\widetilde{J}(z) = \frac{1}{2}\|u_z(T) - \overline{u}\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\langle Dz, z\rangle_\Sigma.$$

Let $h \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ be an arbitrary but given direction for which we have

$$\begin{aligned}
&\widetilde{J}(z+h) - \widetilde{J}(z) \\
&\quad = \frac{1}{2}\,\|u_{z+h}(T) - \overline{u}\|^2_{L_2(\Omega)} - \frac{1}{2}\,\|u_z(T) - \overline{u}\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\langle D(z+h), z+h\rangle_\Sigma - \frac{\alpha}{2}\langle Dz, z\rangle_\Sigma \\
&\quad = \langle u_z(T) - \overline{u}, v(T)\rangle_{L_2(\Omega)} + \frac{1}{2}\|v(T)\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\langle Dz, h\rangle_\Sigma + \frac{\alpha}{2}\langle Dh, z\rangle_\Sigma + \frac{\alpha}{2}\langle Dh, h\rangle_\Sigma,
\end{aligned}$$

and where $v(x,t) := u_{z+h}(x,t) - u_z(x,t)$ is the unique solution of the heat equation

$$\partial_t v(x,t) - \Delta v(x,t) = 0 \quad \text{in } Q, \quad v(x,t) = h(x,t) \quad \text{on } \Sigma, \quad v(x,0) = 0 \quad \text{on } \Omega.$$

By applying Green's second formula for the pair (v,p),

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[p(x,t)\Big(\partial_t - \Delta\Big)v(x,t) + v(x,t)\Big(\partial_t + \Delta\Big)p(x,t)\Big]\,dx\,dt \\
&\qquad = \int_\Omega \Big[v(x,T)p(x,T) - v(x,0)p(x,0)\Big]\,dx \\
&\qquad\qquad + \int_0^T \int_\Gamma \left(\frac{\partial}{\partial n_x}p(x,t)v(x,t) - \frac{\partial}{\partial n_x}v(x,t)p(x,t)\right)\,ds_x\,dt
\end{aligned}$$

we obtain

$$\int_\Omega [u_z(x,T) - \overline{u}(x)]v(x,T)\,dx + \int_\Gamma \frac{\partial}{\partial n_x}p(x,t)h(x,t)\,ds_x\,dt = 0.$$

Therefore, by using the self–adjointness (2.6) of the hypersingular heat boundary integral operator D we conclude

$$\begin{aligned}
&\widetilde{J}(z+h) - \widetilde{J}(z) \\
&\quad = \frac{\alpha}{2}\langle Dz, h\rangle_\Sigma + \frac{\alpha}{2}\langle \kappa_T D \kappa_T z, h\rangle_\Sigma - \langle \partial_n p, h\rangle_\Sigma + \frac{1}{2}\,\|v(T)\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\langle Dh, h\rangle_\Sigma \\
&\quad = \langle \alpha\widetilde{D}z - \partial_n p, h\rangle_\Sigma + \mathcal{O}\left(\|h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}\right),
\end{aligned}$$

where we have used

$$\|v(T)\|_{L_2(\Omega)} \le c\,\|h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}, \quad \langle Dh, h\rangle_\Sigma \le c_2^D\,\|h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

This implies that the gradient of $\widetilde{J}(z)$ satisfies

$$\langle \nabla\widetilde{J}(z), h\rangle_\Sigma = \langle \alpha\widetilde{D}z - \partial_n p, h\rangle_\Sigma\,,$$

from which the assertion follows, see also [4, 7]. ■

In the following we will use a boundary element approach to solve the optimality system, i.e. the primal heat equation (2.2), the adjoint heat equation (2.7), and the optimality condition (2.8).

3 Boundary integral equations

In this section we first recall the boundary integral equations for the heat equation (2.2). Some mapping properties of the standard boundary integral layer heat operators can be found in, e.g., [3]. For the adjoint heat equation (2.7), instead of using the volume potential of the state u, we introduce some boundary potentials with a regular kernel.

3.1 The primal heat equation

Let us first consider the primal heat equation (2.2), where the solution is given by the representation formula for $(\widetilde{x}, t) \in Q$,

$$\begin{aligned} u(\widetilde{x}, t) &= \int_0^t \int_\Gamma \mathcal{E}(\widetilde{x} - y, t - \tau) \frac{\partial}{\partial n_y} u(y, \tau)\, ds_y\, d\tau \\ &\quad - \int_0^t \int_\Gamma \frac{\partial}{\partial n_y} \mathcal{E}(\widetilde{x} - y, t - \tau) z(y, \tau)\, ds_y\, d\tau + \int_\Omega \mathcal{E}(\widetilde{x} - y, t) u_0(y)\, dy, \end{aligned}$$

where $\mathcal{E}(x, t)$ is the fundamental solution of the heat equation as given in (2.4). By taking the limit $\Omega \ni \widetilde{x} \to x \in \Gamma$, we obtain the first kind boundary integral equation to find $\omega(x, t) := \partial_n u(x, t)$ such that

$$(V\omega)(x, t) = (\frac{1}{2} I + K) z(x, t) - (M_0 u_0)(x, t) \quad \text{for } (x, t) \in \Sigma. \tag{3.1}$$

Here,

$$(V\omega)(x, t) = \int_0^t \int_\Gamma \mathcal{E}(x - y, t - \tau) \omega(y, \tau)\, ds_y\, d\tau \quad \text{for } (x, t) \in \Sigma$$

is the single layer heat boundary integral operator $V : H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2}, \frac{1}{4}}(\Sigma)$, and

$$(Kz)(x, t) = \int_0^t \int_\Gamma \frac{\partial}{\partial n_y} \mathcal{E}(x - y, t - \tau) z(y, \tau)\, ds_y\, d\tau \quad \text{for } (x, t) \in \Sigma$$

is the double layer heat boundary integral operator $K : H^{\frac{1}{2}, \frac{1}{4}}(\Sigma) \to H^{\frac{1}{2}, \frac{1}{4}}(\Sigma)$, see [3]. Moreover,

$$(M_0 u_0)(x, t) = \int_\Omega \mathcal{E}(x - y, t) u_0(y)\, dy \quad \text{for } (x, t) \in \Sigma$$

is the related Newton potential. Since the single layer heat boundary integral operator V is $H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma)$-elliptic, i.e.,

$$\langle V\omega, \omega \rangle_\Sigma \geq c_1^V \|\omega\|^2_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma)} \quad \text{for all } \omega \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma),$$

the boundary integral equation (3.1) is solvable, when a Dirichlet datum z is given, and we obtain

$$\omega = V^{-1}(\frac{1}{2}I + K)z - V^{-1}M_0u_0. \tag{3.2}$$

3.2 The adjoint heat equation

Next we consider the adjoint heat equation (2.7). Since the time reversal of the adjoint state variable, $\kappa_T p$, is a solution of the heat equation, i.e.,

$$\begin{aligned}
\partial_t(\kappa_T p)(x,t) - \Delta(\kappa_T p)(x,t) &= 0 && \text{for } (x,t) \in Q,\\
\kappa_T p(x,t) &= 0 && \text{for } (x,t) \in \Sigma,\\
\kappa_T p(x,0) &= u(x,T) - \overline{u}(x) && \text{for } x \in \Omega,
\end{aligned}$$

the representation formula for $(\widetilde{x}, t) \in Q$ gives

$$\kappa_T p(\widetilde{x}, t) = \int_0^t \int_\Gamma \mathcal{E}(\widetilde{x} - y, t - \tau) \frac{\partial}{\partial n_y} \kappa_T p(y, \tau)\, ds_y\, d\tau + \int_\Omega \mathcal{E}(\widetilde{x} - y, t) \kappa_T p(y, 0)\, dy, \tag{3.3}$$

and therefore the first kind boundary integral equation

$$(V(\kappa_T q))(x,t) = (M_0 \overline{u})(x,t) - (M_0 u(\cdot, T))(x,t) \quad \text{for } (x,t) \in \Sigma \tag{3.4}$$

to determine the unknown Neumann datum $q(x,t) = \partial_n p(x,t)$ for $(x,t) \in \Sigma$ follows.

In (3.4), the unknown state $u(\cdot, T)$ at the final time T appears in the Newton potential. Hence, in what follows we will modify the representation formula (3.3). The crucial idea is to use an auxiliary function

$$G(x,t,\tau) = \left(\frac{t}{T + t - \tau} \right)^{d/2} e^{\frac{T-\tau}{T+t-\tau} \frac{|x|^2}{4t}} \quad \text{for } x \in \Omega;\ t, \tau \in (0,T), \tag{3.5}$$

which satisfies

$$\mathcal{E}(x,t) G(x,t,\tau) = \mathcal{E}(x, T + t - \tau), \qquad \lim_{\tau \to T^-} G(\widetilde{x} - y, t, \tau) = 1 \quad \text{for all } t \in (0,T).$$

For $y \in \Omega$ we then have

$$\begin{aligned}
u(y,T) &= u(y,T)G(\widetilde{x} - y, t, T) - u(y,0)G(\widetilde{x} - y, t, 0) + u_0(y)G(\widetilde{x} - y, t, 0)\\
&= \int_0^T \partial_\tau \Big[u(y,\tau) G(\widetilde{x} - y, t, \tau) \Big]\, d\tau + u_0(y) G(\widetilde{x} - y, t, 0)\\
&= \int_0^T \partial_\tau G(\widetilde{x} - y, t, \tau) u(y,\tau)\, d\tau + \int_0^T G(\widetilde{x} - y, t, \tau) \partial_\tau u(y,\tau)\, d\tau + u_0(y) G(\widetilde{x} - y, t, 0).
\end{aligned}$$

Hence we can write the Newton potential in the representation formula (3.3) as

$$
\begin{aligned}
(M_0 u(\cdot,T))(\widetilde{x},t) &= \int_\Omega \mathcal{E}(\widetilde{x}-y,t)u(y,T)\,dy \\
&= \int_0^T \int_\Omega \mathcal{E}(\widetilde{x}-y,t)\partial_\tau G(\widetilde{x}-y,t,\tau)u(y,\tau)\,dy\,d\tau \\
&\quad + \int_0^T \int_\Omega \mathcal{E}(\widetilde{x}-y,t)G(\widetilde{x}-y,t,\tau)\Delta_y u(y,\tau)\,dy\,d\tau \\
&\quad + \int_\Omega \mathcal{E}(\widetilde{x}-y,t)G(\widetilde{x}-y,t,0)u_0(y)\,dy.
\end{aligned}
$$

It is easy to check that

$$
\Delta_y[\mathcal{E}(\widetilde{x}-y,t)G(\widetilde{x}-y,t,\tau)] = \Delta_y \mathcal{E}(\widetilde{x}-y,T+t-\tau) = -\mathcal{E}(\widetilde{x}-y,t)\partial_\tau G(\widetilde{x}-y,t,\tau),
$$

and by definition, we have

$$
\mathcal{E}(\widetilde{x}-y,t)G(\widetilde{x}-y,t,\tau) = \mathcal{E}(\widetilde{x}-y,T+t-\tau).
$$

Together with Green's second formula we finally obtain

$$
\begin{aligned}
(M_0 u(\cdot,T))(\widetilde{x},t) &= \int_0^T \int_\Omega \mathcal{E}(\widetilde{x}-y,T+t-\tau)\Delta_y u(y,\tau)\,dy\,d\tau \\
&\quad - \int_0^T \int_\Omega u(y,\tau)\Delta_y \mathcal{E}(\widetilde{x}-y,T+t-\tau)\,dy\,d\tau + \int_\Omega \mathcal{E}(\widetilde{x}-y,T+t)u_0(y)\,dy \\
&= \int_0^T \int_\Gamma \mathcal{E}(\widetilde{x}-y,T+t-\tau)\frac{\partial}{\partial n_y}u(y,\tau)\,ds_y\,d\tau \\
&\quad - \int_\Gamma u(y,\tau)\frac{\partial}{\partial n_y}\mathcal{E}(\widetilde{x}-y,T+t-\tau)\,ds_y\,d\tau + \int_\Omega \mathcal{E}(\widetilde{x}-y,T+t)u_0(y)\,dy,
\end{aligned}
$$

and this gives a modified representation formula for the adjoint variable for $(\widetilde{x},t) \in Q$,

$$
\begin{aligned}
\kappa_T p(\widetilde{x},t) &= \int_0^t \int_\Gamma \mathcal{E}(\widetilde{x}-y,t-\tau)\frac{\partial}{\partial n_y}\kappa_T p(y,\tau)\,ds_y\,d\tau - \int_\Omega \mathcal{E}(\widetilde{x}-y,t)\overline{u}(y)\,dy \\
&\quad + \int_0^T \int_\Gamma \left[\mathcal{E}(\widetilde{x}-y,T+t-\tau)\frac{\partial}{\partial n_y}u(y,\tau) - u(y,\tau)\frac{\partial}{\partial n_y}\mathcal{E}(\widetilde{x}-y,T+t-\tau)\right] ds_y\,d\tau \\
&\quad + \int_\Omega \mathcal{E}(\widetilde{x}-y,T+t)u_0(y)\,dy.
\end{aligned}
\tag{3.6}
$$

By taking the limit $\Omega \ni \widetilde{x} \to x \in \Gamma$ we obtain a boundary integral equation for $(x,t) \in \Sigma$,

$$
0 = (V(\kappa_T q))(x,t) + (V_1\omega)(x,t) - (K_1 z)(x,t) - (\widetilde{M_0}\overline{u})(x,t) + (M_{10}u_0)(x,t), \tag{3.7}
$$

where

$$\begin{aligned}
(V_1\omega)(x,t) &= \int_0^T \int_\Gamma \mathcal{E}(x-y,T+t-\tau)\omega(y,\tau)\,ds_y\,d\tau,\\
(K_1 z)(x,t) &= \int_0^T \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,T+t-\tau)z(y,\tau)\,ds_y\,d\tau
\end{aligned}$$

are the bi–single and the bi–double layer heat boundary integral operators, respectively, defined for $(x,t)\in\Sigma$. In addition to

$$(\widetilde{M_0}\overline{u})(x) = \int_\Omega \mathcal{E}(x-y,t)\overline{u}(y)\,dy \quad \text{for } (x,t)\in\Sigma$$

we introduce the volume potential

$$(M_{10}u_0)(x,t) = \int_\Omega \mathcal{E}(x-y,T+t)u_0(y)dy \quad \text{for } (x,t)\in\Sigma.$$

When inserting (3.2) into the boundary integral equation (3.7), this gives

$$V(\kappa_T q) = K_1 z - V_1 V^{-1}(\frac{1}{2}I+K)z + V_1 V^{-1}M_0 u_0 + \widetilde{M_0}\overline{u} - M_{10}u_0,$$

and hence we conclude

$$\kappa_T q = V^{-1}K_1 z - V^{-1}V_1 V^{-1}(\frac{1}{2}I+K)z + V^{-1}V_1V^{-1}M_0u_0 + V^{-1}M_0\overline{u} - V^{-1}M_{10}u_0. \quad (3.8)$$

Now the optimality condition (2.8) can be rewritten as a variational inequality to find $z\in\mathcal{U}_{ad}$, such that

$$\langle \mathcal{T}_\alpha z - g, w - z\rangle_\Sigma \geq 0 \quad \text{for all } w\in\mathcal{U}_{ad}, \quad (3.9)$$

where

$$\mathcal{T}_\alpha := \alpha\widetilde{D} - \kappa_T V^{-1}K_1 + \kappa_T V^{-1}V_1V^{-1}(\frac{1}{2}I+K) \quad (3.10)$$

and

$$g := \kappa_T V^{-1}M_0\overline{u} + \kappa_T\left(V^{-1}V_1V^{-1}M_0 - V^{-1}M_{10}\right)u_0. \quad (3.11)$$

3.3 Mapping properties

To investigate the properties of the composed boundary integral operator $\mathcal{T}_\alpha$ as defined in (3.10), let us summarize some properties of the bi–layer heat boundary integral operators V_1 and K_1 which are similar to the properties of the Bi–Laplace boundary integral operators as considered in [10, 12].

Lemma 3.1 *For $\omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ we have*

$$\langle (\frac{1}{2}I + K')\omega, \kappa_T V_1 \omega\rangle_\Sigma - \langle \kappa_T K_1' \omega, V\omega\rangle_\Sigma = \|(\widetilde{V}\omega)(\cdot, T)\|^2_{L_2(\Omega)} \tag{3.12}$$

with the single layer heat potential

$$(\widetilde{V}\omega)(x,t) = \int_0^t \int_\Gamma \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\, ds_y\, d\tau \quad \textit{for } (x,t) \in Q,$$

and with the adjoint double layer heat potentials

$$\begin{aligned} (K'\omega)(x,t) &= \int_0^t \int_\Gamma \frac{\partial}{\partial n_x} \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\, ds_y\, d\tau \quad \textit{for } (x,t) \in \Sigma, \\ (K_1'\omega)(x,t) &= \int_0^T \int_\Gamma \frac{\partial}{\partial n_x} \mathcal{E}(x-y, T+t-\tau)\omega(y,\tau)\, ds_y\, d\tau \quad \textit{for } (x,t) \in \Sigma, \end{aligned}$$

which satisfy

$$\langle \kappa_T \omega, Kz\rangle_\Sigma = \langle \kappa_T z, K'\omega\rangle_\Sigma, \quad \langle \kappa_T \omega, K_1 z\rangle_\Sigma = \langle \kappa_T z, K_1'\omega\rangle_\Sigma. \tag{3.13}$$

Proof. Consider the following functions for $(x,t) \in Q$

$$\begin{aligned} u(x,t) = (\widetilde{V}\omega)(x,t) &= \int_0^t \int_\Gamma \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\, ds_y\, d\tau, \\ v(x,t) &= \int_0^T \int_\Gamma \mathcal{E}(x-y, 2T-t-\tau)\omega(y,\tau)\, ds_y\, d\tau \end{aligned}$$

which are solutions of the heat equation and of the adjoint heat equation, respectively,

$$\partial_t u(x,t) - \Delta u(x,t) = 0, \quad -\partial_t v(x,t) - \Delta v(x,t) = 0 \quad \text{for } (x,t) \in Q.$$

Moreover, we have

$$u(x,T) = v(x,T), \quad u(x,0) = 0.$$

The application of the standard interior Dirichlet and Neumann trace operators γ_0 and γ_1 gives

$$\begin{aligned} \gamma_0 u(x,t) &= (V\omega)(x,t), & \gamma_1 u(x,t) &= (\frac{1}{2}I + K')\omega(x,t), \\ \gamma_0 v(x,t) &= \kappa_T(V_1\omega)(x,t), & \gamma_1 v(x,t) &= \kappa_T(K_1'\omega)(x,t). \end{aligned}$$

Now the assertion follows from Green's second formula, i.e.

$$\begin{aligned} &\int_0^T \int_\Omega \Big[v(x,t)\Big(\partial_t - \Delta\Big)u(x,t) + u(x,t)\Big(\partial_t + \Delta\Big)v(x,t)\Big]\, dx\, dt \\ &\qquad = \int_0^T \int_\Gamma \Big[\gamma_1 v(x,t)\gamma_0 u(x,t) - \gamma_1 u(x,t)\gamma_0 v(x,t)\Big]\, ds_x\, dt \\ &\qquad\qquad + \int_\Omega [u(x,T)v(x,T) - u(x,0)v(x,0)]\, dx. \end{aligned}$$

■

As in the case of the Bi–Laplace operator, see [10, 12], we can state the following properties.

Lemma 3.2 *For the layer and bi–layer heat boundary integral operators, there hold the relations*

$$KV = VK', \quad DK = K'D, \quad VD = \frac{1}{4}I - K^2, \quad DV = \frac{1}{4}I - K'^2, \tag{3.14}$$

$$V^{-1}(\frac{1}{2}I + K) = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K), \tag{3.15}$$

and

$$\begin{aligned} VK_1' + V_1K' &= K_1V + KV_1, & (3.16)\\ DK_1 - D_1K &= K_1'D - K'D_1, & (3.17)\\ V_1D - VD_1 + KK_1 + K_1K &= 0, & (3.18)\\ DV_1 - D_1V + K'K_1' + K_1'K' &= 0. & (3.19) \end{aligned}$$

Note that D_1 is the normal derivative of the bi–double layer heat potential, i.e.

$$(D_1z)(x,t) = \frac{\partial}{\partial n_x}\int_0^T \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y, T+t-\tau)z(y,\tau)\, ds_y\, d\tau, \quad (x,t) \in \Sigma. \tag{3.20}$$

Proof. The relations of (3.14) for the layer heat boundary integral operators are well known, see [3]. The relation (3.15) is an alternative representation of the so–called Dirichlet to Neumann operator, see [13] for similar properties of the Laplace boundary integral operators. Indeed, by (3.14) we have

$$\begin{aligned} D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) &= V^{-1}(\frac{1}{4}I - K^2) + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\\ &= V^{-1}(\frac{1}{2}I - K)(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\\ &= (\frac{1}{2}V^{-1} - K'V^{-1})(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\\ &= V^{-1}(\frac{1}{2}I + K). \end{aligned}$$

To establish the relations (3.16)–(3.19), let $\omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$, $\varphi \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ be arbitrary. We then define, for $(x,t) \in Q$,

$$\begin{aligned} u(x,t) &= \int_0^T \int_\Gamma \mathcal{E}(x-y, T+t-\tau)\omega(y,\tau)\, ds_y\, d\tau\\ &\qquad + \int_0^T \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y, T+t-\tau)\varphi(y,\tau)\, ds_y\, d\tau,\\ v(x,t) &= \int_0^t \int_\Gamma \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\, ds_y\, d\tau + \int_0^t \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y, t-\tau)\varphi(y,\tau)\, ds_y\, d\tau, \end{aligned}$$

which are solutions of the heat equation. Their related boundary and initial conditions are given by

$$\gamma_0 u = V_1\omega + K_1\varphi, \qquad \gamma_1 u = K_1'\omega + D_1\varphi, \qquad u(x,0) = v(x,T),$$
$$\gamma_0 v = V\omega + (-\frac{1}{2}I + K)\varphi, \qquad \gamma_1 v = (\frac{1}{2}I + K')\omega - D\varphi, \qquad v(x,0) = 0.$$

Moreover, by using $u(x,0) = v(x,T)$ we can also represent the function $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \int_0^t \int_\Gamma \mathcal{E}(x-y,t-\tau)\gamma_1 u(y,\tau)\, ds_y\, d\tau - \int_0^t \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,t-\tau)\gamma_0 u(y,\tau)\, ds_y\, d\tau \\ &\quad + \int_\Omega \mathcal{E}(x-y,t)v(y,T)\, dy. \end{aligned}$$

Again, we can modify the volume potential as before to obtain the representation formula

$$\begin{aligned} u(x,t) = &\int_0^t \int_\Gamma \mathcal{E}(x-y,t-\tau)\gamma_1 u(y,\tau)\, ds_y\, d\tau - \int_0^t \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,t-\tau)\gamma_0 u(y,\tau)\, ds_y\, d\tau \\ + &\int_0^T \int_\Gamma \mathcal{E}(x-y,T+t-\tau)\gamma_1 v(y,\tau)\, ds_y\, d\tau - \int_0^T \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,T+t-\tau)\gamma_0 v(y,\tau)\, ds_y\, d\tau. \end{aligned}$$

Hence, by taking the Dirichlet and Neumann traces we conclude

$$\begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & -D_1 & K_1' \end{pmatrix} \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \\ \gamma_0 v \\ \gamma_1 v \end{pmatrix},$$

and by inserting the traces of u and v we obtain

$$\begin{aligned} V_1\omega + K_1\varphi &= (\frac{1}{2}I - K)[V_1\omega + K_1\varphi] + V[K_1'\omega + D_1\varphi] \\ &\quad -K_1[V\omega + (-\frac{1}{2}I + K)\varphi] + V_1[(\frac{1}{2}I + K')\omega - D\varphi], \end{aligned}$$

and

$$\begin{aligned} K_1'\omega + D_1\varphi &= D[V_1\omega + K_1\varphi] + (\frac{1}{2}I + K')[K_1'\omega + D_1\varphi] \\ &\quad -D_1[V\omega + (-\frac{1}{2}I + K)\varphi] + K_1'[(\frac{1}{2}I + K')\omega - D\varphi], \end{aligned}$$

which hold for all $\omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$, $\varphi \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, and which imply

$$\begin{aligned} V_1 &= (\frac{1}{2}I - K)V_1 + VK_1' - K_1V + V_1(\frac{1}{2}I + K'), \\ K_1 &= (\frac{1}{2}I - K)K_1 + VD_1 + K_1(\frac{1}{2}I - K) - V_1D, \\ K_1' &= DV_1 + (\frac{1}{2}I + K')K_1' - D_1V + K_1'(\frac{1}{2}I + K'), \\ D_1 &= DK_1 + (\frac{1}{2}I + K')D_1 + D_1(\frac{1}{2}I - K) - K_1'D. \end{aligned}$$

Now the assertion follows. ■

Note that the boundary integral operators V, D, V_1, and D_1 are self–adjoint with respect to the time–twisted duality pairing, see (2.6) for D, and hence the operator $\widetilde{D}$ is self–adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_\Sigma$, i.e., for all $\omega,\theta \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$, $\varphi, w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ we have

$$\langle V\omega, \kappa_T\theta\rangle_\Sigma = \langle V\theta, \kappa_T\omega\rangle_\Sigma, \qquad \langle V_1\omega, \kappa_T\theta\rangle_\Sigma = \langle V_1\theta, \kappa_T\omega\rangle_\Sigma, \tag{3.21}$$

$$\langle D_1\varphi, \kappa_T w\rangle_\Sigma = \langle D_1 w, \kappa_T\varphi\rangle_\Sigma, \qquad \langle \widetilde{D}\varphi, w\rangle_\Sigma = \langle \widetilde{D}w, \varphi\rangle_\Sigma. \tag{3.22}$$

Lemma 3.3 *The operator*

$$A := \begin{pmatrix} V_1 & -K_1 \\ -K_1' & D_1 \end{pmatrix},$$

satisfies

$$\langle A\begin{pmatrix}\omega\\ \varphi\end{pmatrix}, \kappa_T\begin{pmatrix}\omega\\ \varphi\end{pmatrix}\rangle_\Sigma = \langle V_1\omega, \kappa_T\omega\rangle_\Sigma - \langle K_1\varphi, \kappa_T\omega\rangle_\Sigma - \langle K_1'\omega, \kappa_T\varphi\rangle_\Sigma + \langle D_1\varphi, \kappa_T\varphi\rangle_\Sigma \geq 0$$

for all $\omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$, $\varphi \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$.

Proof. For $\omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and $\varphi \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ we define, for $(x,t) \in \mathbb{R}^d\backslash\Gamma \times (0,T)$,

$$u(x,t) = \int_0^T\int_\Gamma \mathcal{E}(x-y, T+t-\tau)\omega(y,\tau)\,ds_y\,d\tau - \int_0^T\int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,T+t-\tau)\varphi(y,\tau)\,ds_y\,d\tau,$$

$$v(x,t) = \int_0^t\int_\Gamma \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\,ds_y\,d\tau - \int_0^t\int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y,t-\tau)\varphi(y,\tau)\,ds_y\,d\tau,$$

which are solutions of the heat equation in both the interior and exterior domains. The related boundary traces of u are given by

$$\gamma_0 u = V_1\omega - K_1\varphi, \quad \gamma_1 u = K_1'\omega - D_1\varphi,$$

while v satisfies jump relations across Σ,

$$[\gamma_0 v] := \gamma_0^{\text{ext}}v - \gamma_0^{\text{int}}v = -\varphi, \quad [\gamma_1 v] := \gamma_1^{\text{ext}}v - \gamma_1^{\text{int}}v = -\omega.$$

Hence we can rewrite the bilinear form of the operator A as

$$\begin{aligned}
\langle A\begin{pmatrix}\omega\\ \varphi\end{pmatrix}, \kappa_T\begin{pmatrix}\omega\\ \varphi\end{pmatrix}\rangle_\Sigma &= \langle\begin{pmatrix}\gamma_0 u\\ -\gamma_1 u\end{pmatrix}, \kappa_T\begin{pmatrix}-[\gamma_1 v]\\ -[\gamma_0 v]\end{pmatrix}\rangle_\Sigma \\
&= \langle \gamma_0 u, \kappa_T\gamma_1^{\text{int}}v\rangle_\Sigma - \langle\gamma_0 u, \kappa_T\gamma_1^{\text{ext}}v\rangle_\Sigma + \langle\gamma_1 u, \kappa_T\gamma_0^{\text{ext}}v\rangle_\Sigma - \langle\gamma_1 u, \kappa_T\gamma_0^{\text{int}}v\rangle_\Sigma \\
&= \langle \gamma_0^{\text{int}} u, \kappa_T\gamma_1^{\text{int}}v\rangle_\Sigma - \langle\gamma_1^{\text{int}} u, \kappa_T\gamma_0^{\text{int}}v\rangle_\Sigma + \langle\gamma_1^{\text{ext}} u, \kappa_T\gamma_0^{\text{ext}}v\rangle_\Sigma - \langle\gamma_0^{\text{ext}} u, \kappa_T\gamma_1^{\text{ext}}v\rangle_\Sigma.
\end{aligned}$$

The application of Green's second formula to the solutions u and v of the heat equation both in the interior and exterior domains gives

$$\langle \gamma_0^{\text{int}} u, \kappa_T \gamma_1^{\text{int}} v\rangle_\Sigma - \langle \gamma_1^{\text{int}} u, \kappa_T \gamma_0^{\text{int}} v\rangle_\Sigma = \int_\Omega u(x,0) v(x,T)\, dx = \int_\Omega [u(x,0)]^2\, dx,$$

and

$$\begin{aligned}&\langle \gamma_1^{\text{ext}} u, \kappa_T \gamma_0^{\text{ext}} v\rangle_\Sigma - \langle \gamma_0^{\text{ext}} u, \kappa_T \gamma_1^{\text{ext}} v\rangle_\Sigma \\ &\qquad = \int_{\Omega_R^c} [u(x,0)]^2\, dx - \langle \gamma_1 v, \kappa_T \gamma_0 u\rangle_{\partial B_R \times I} + \langle \gamma_1 u, \kappa_T \gamma_0 v\rangle_{\partial B_R \times I},\end{aligned}$$

where $B_R := \{x \in \mathbb{R}^d : |x| < R\}$ is a sufficiently large ball containing Ω, and $\Omega_R^c := B_R \setminus \overline{\Omega}$, $Q_R^c := \Omega_R^c \times I$. Thus we obtain

$$\langle A \begin{pmatrix}\omega\\ \varphi\end{pmatrix}, \kappa_T \begin{pmatrix}\omega\\ \varphi\end{pmatrix}\rangle_\Sigma = \int_{\Omega\cup\Omega_R^c} [u(x,0)]^2\, dx - \int_{\partial B_R \times I} \kappa_T u \partial_r v\, ds_x\, dt + \int_{\partial B_R \times I} \kappa_T v \partial_r u\, ds_x\, dt.$$

We will show that the last two terms tend to zero as $R \to \infty$. To do this, let us consider the function v first. Let us choose $0 < R_0 < R$ such that $\overline{\Omega} \subset B_{R_0}$. By the representation formula for the solution v of the heat equation, it follows that outside $Q_{R_0}^c$, in particular for $|x| > R_0$, the function v coincides with

$$v_0(x,t) := \int_0^t \int_{\partial B_{R_0}} \mathcal{E}(x-y, t-\tau) \omega_0(y,\tau)\, ds_y\, d\tau - \int_0^t \int_{\partial B_{R_0}} \frac{\partial}{\partial n_y} \mathcal{E}(x-y, t-\tau) \varphi_0(y,\tau)\, ds_y\, d\tau,$$

where the single and the double layer potentials are now defined for density functions on $\Sigma_{R_0} := \partial B_{R_0} \times I$, i.e. $\omega_0 := \partial_r v|_{\Sigma_{R_0}}$, $\varphi_0 := v|_{\Sigma_{R_0}}$. The densities ω_0, φ_0, as well as the boundary Σ_{R_0}, are smooth. We can now easily estimate v, and $\partial_r v$ on the boundary Σ_R for $R > R_0$, using the behaviour of the fundamental solution $\mathcal{E}(x,t)$. From the simple estimates, for all $\mu \in \mathbb{R}$,

$$|\mathcal{E}(x,t)| \le c_\mu\, t^{-\mu}\, |x|^{2\mu - d}, \quad |\nabla \mathcal{E}(x,t)| \le c_\mu\, t^{-\mu} |x|^{2\mu-d-1}$$

and

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \mathcal{E}(x,t) \right| \le c_\mu\, t^{-\mu}\, |x|^{2\mu-d-2}, \quad i,j = 1,2,$$

we obtain for finite T,

$$\kappa_T v = \mathcal{O}(R^{-d}), \quad \partial_r v = \mathcal{O}(R^{-d-1}) \quad \text{as} \quad |x| = R \to \infty.$$

Similarly, for $(x,t) \in \partial B_R \times I$, the kernel $\mathcal{E}(x-y, T+t-\tau)$, $(y,\tau) \in \Sigma$, is smooth. Then $\kappa_T u$ and $\partial_r u$ are bounded as $|x| = R \to \infty$. Hence

$$-\int_{\partial B_R \times I} \kappa_T u \partial_r v\, ds_x\, dt + \int_{\partial B_R \times I} \kappa_T v \partial_r u\, ds_x\, dt = \mathcal{O}(R^{-1}) \to 0 \quad \text{as} \quad |x| = R \to \infty.$$

Hence we finally conclude

$$\langle A \begin{pmatrix} \omega \\ \varphi \end{pmatrix}, \kappa_T \begin{pmatrix} \omega \\ \varphi \end{pmatrix} \rangle_\Sigma = \int_{\mathbb{R}^d} [u(x,0)]^2 \, dx \geq 0.$$

■

Corollary 3.4 *The boundary integral operators V_1 and D_1 are positive semi–definite with respect to the time–twisted duality pairing $\langle \cdot, \kappa_T \cdot \rangle_\Sigma$, i.e. we have*

$$\langle V_1 \omega, \kappa_T \omega \rangle_\Sigma \geq 0 \quad \textit{for all } \omega \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), \quad \langle D_1 \varphi, \kappa_T \varphi \rangle_\Sigma \geq 0 \quad \textit{for all } \varphi \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

To close this section, let us recall the mapping properties of the Newton potential M_0, see [9, Lemma 7.10].

Lemma 3.5 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be bounded. Then, for any $f \in L_2(\Omega)$ here holds*

$$\|\widetilde{M_0} f\|_{H^{1,\frac{1}{2}}(Q)} \leq C(\Omega) \, \|f\|_{L_2(\Omega)}.$$

Note that, since $\mathcal{E}(x, T+t) \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ for $T > 0$, the operator M_{10} is continuous on the considered Sobolev spaces.

We are now in a position to prove the properties of the boundary integral operator $\mathcal{T}_\alpha$.

Theorem 3.6 *The operator $\mathcal{T}_\alpha : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ as defined in* (3.10)*,*

$$\mathcal{T}_\alpha = \alpha \widetilde{D} - \kappa_T V^{-1} K_1 + \kappa_T V^{-1} V_1 V^{-1} (\frac{1}{2} I + K)$$

is bounded, self–adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Sigma$ and $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$–elliptic, i.e. there exists a constant $c_1^{\mathcal{T}_\alpha} > 0$ such that

$$\langle \mathcal{T}_\alpha z, z \rangle_\Sigma \geq c_1^{\mathcal{T}_\alpha} \|z\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Proof. The proof is similar to the proof of Theorem 4.4 in [10], hence we skip the details. Note that, for the mapping properties of the layer heat potentials, see [3]. In particular, we have

$$V^{-1} : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), \quad K : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \quad \widetilde{D} : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

Since the kernels of the bi–layer heat potentials are regular, we can also derive

$$V_1 : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \quad K_1 : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

■

Hence we conclude the unique solvability of the variational inequality (3.9).

4 Symmetric boundary element approximations

In this section, we investigate a symmetric boundary integral formulation by using also a second boundary integral equation for the solution of the adjoint heat boundary value problem. We ensure unique solvability and we derive a priori error estimates for a Galerkin boundary element approximation.

In particular, when computing the normal derivative of the representation formula (3.6) of the adjoint variable p, this gives for all $(x,t) \in \Sigma$

$$\kappa_T q(x,t) = (\frac{1}{2}I + K')\kappa_T q(x,t) + (K_1'\omega)(x,t) - (D_1 z)(x,t) - (M_1\overline{u})(x,t) + (M_{11}u_0)(x,t), \tag{4.1}$$

where, in addition, we introduce the Newton potentials for $(x,t) \in \Sigma$

$$(M_1\overline{u})(x,t) = \frac{\partial}{\partial n_x}\int_\Omega \mathcal{E}(x-y,t)\overline{u}(y)\,dy, \quad (M_{11}u_0)(x,t) = \frac{\partial}{\partial n_x}\int_\Omega \mathcal{E}(x-y,T+t)u_0(y)\,dy.$$

By substituting (3.2) and (3.8) into the right hand side of (4.1) we obtain the alternative representation

$$\begin{aligned}\kappa_T q \;=\;& (\frac{1}{2}I + K')V^{-1}K_1 z - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K)z + (\frac{1}{2}I + K')V^{-1}V_1V^{-1}M_0u_0\\ &+(\frac{1}{2}I + K')V^{-1}M_0\overline{u} - (\frac{1}{2}I + K')V^{-1}M_{10}u_0 + K_1'V^{-1}(\frac{1}{2}I + K)z - K_1'V^{-1}M_0u_0\\ &-D_1 z - M_1\overline{u} + M_{11}u_0.\end{aligned}$$

Hence we have to solve the variational inequality to find the control $z \in \mathcal{U}_{\rm ad}$ such that

$$\langle \mathcal{T}_\alpha z - g, w - z\rangle_\Sigma \geq 0 \quad \text{for all } w \in \mathcal{U}_{\rm ad}, \tag{4.2}$$

where

$$\begin{aligned}\mathcal{T}_\alpha \;=\;& \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T K_1'V^{-1}(\frac{1}{2}I + K) - \kappa_T(\frac{1}{2}I + K')V^{-1}K_1\\ &+\kappa_T(\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K)\end{aligned} \tag{4.3}$$

is an alternative representation of $\mathcal{T}_\alpha$ as defined in (3.10), and

$$\begin{aligned}\kappa_T g \;=\;& (\frac{1}{2}I + K')V^{-1}M_0\overline{u} - M_1\overline{u} + (\frac{1}{2}I + K')V^{-1}V_1V^{-1}M_0u_0\\ &-(\frac{1}{2}I + K')V^{-1}M_{10}u_0 - K_1'V^{-1}M_0u_0 + M_{11}u_0\end{aligned} \tag{4.4}$$

is the related right hand side.

Theorem 4.1 *The operator $\mathcal{T}_\alpha$ as given in (4.3) coincides with the operator as defined in* (3.10). *In particular, $\mathcal{T}_\alpha$ is bounded, self–adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_\Sigma$, and $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$–elliptic, i.e., there holds, for some $c_1^{\mathcal{T}_\alpha} > 0$,*

$$\langle \mathcal{T}_\alpha z, z\rangle_\Sigma \geq c_1^{\mathcal{T}_\alpha}\|z\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Proof. The self–adjointness of $\mathcal{T}_\alpha$ is obvious from the symmetric representation (4.3) and (3.13), (3.21), (3.22). In particular, the operators $\mathcal{T}_\alpha$ in the symmetric representation (4.3) and in the non–symmetric representation (3.10) coincide. Indeed, by using (3.14) and (3.16) we have

$$\begin{aligned}
\mathcal{T}_\alpha &= \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T\Big(K_1' - (\frac{1}{2}I + K')V^{-1}V_1\Big)V^{-1}(\frac{1}{2}I + K) - \kappa_T(\frac{1}{2}I + K')V^{-1}K_1 \\
&= \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T V^{-1}\Big(VK_1' - KV_1 - \frac{1}{2}V_1\Big)V^{-1}(\frac{1}{2}I + K) - \kappa_T(\frac{1}{2}I + K')V^{-1}K_1 \\
&= \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T V^{-1}\Big(K_1 V - V_1 K' - \frac{1}{2}V_1\Big)V^{-1}(\frac{1}{2}I + K) - \kappa_T(\frac{1}{2}I + K')V^{-1}K_1 \\
&= \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T V^{-1}K_1(\frac{1}{2}I + K) + \kappa_T V^{-1}V_1(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) \\
&\qquad -\kappa_T(\frac{1}{2}I + K')V^{-1}K_1.
\end{aligned}$$

By using (3.15), (3.14) and (3.18) we further conclude

$$\begin{aligned}
\mathcal{T}_\alpha &= \alpha\widetilde{D} + \kappa_T D_1 - \kappa_T V^{-1}K_1(\frac{1}{2}I + K) + \kappa_T V^{-1}V_1\Big(V^{-1}(\frac{1}{2}I + K) - D\Big) \\
&\qquad -\kappa_T(\frac{1}{2}I + K')V^{-1}K_1 \\
&= \alpha\widetilde{D} + \kappa_T V^{-1}\Big(VD_1 - K_1(\frac{1}{2}I + K) + V_1V^{-1}(\frac{1}{2}I + K) - V_1 D - (\frac{1}{2}I + K)K_1\Big) \\
&= \alpha\widetilde{D} + \kappa_T V^{-1}\Big(V_1V^{-1}(\frac{1}{2}I + K) - K_1\Big) \\
&= \alpha\widetilde{D} - \kappa_T V^{-1}K_1 + \kappa_T V^{-1}V_1V^{-1}(\frac{1}{2}I + K)
\end{aligned}$$

and we obtain the non–symmetric representation (3.10).

Moreover, the ellipticity estimate can be shown directly by using Lemma 3.3. Indeed, for $z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and $\omega = V^{-1}(\frac{1}{2}I + K)z \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ we have

$$\begin{aligned}
\langle \mathcal{T}_\alpha z, z\rangle_\Sigma &= \alpha\langle \widetilde{D}z, z\rangle_\Sigma + \langle \kappa_T D_1 z, z\rangle_\Sigma - \langle \kappa_T K_1'\omega, z\rangle_\Sigma \\
&\quad -\langle \kappa_T(\frac{1}{2}I + K')V^{-1}K_1 z, z\rangle_\Sigma + \langle \kappa_T(\frac{1}{2}I + K')V^{-1}V_1\omega, z\rangle_\Sigma \\
&= \alpha\langle \widetilde{D}z, z\rangle_\Sigma + \langle \kappa_T D_1 z, z\rangle_\Sigma - \langle K_1'\omega, \kappa_T z\rangle_\Sigma - \langle K_1 z, \kappa_T\omega\rangle_\Sigma + \langle V_1\omega, \kappa_T\omega\rangle_\Sigma \\
&\geq \alpha\langle \widetilde{D}z, z\rangle_\Sigma \geq \alpha c_1^D \, \|z\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.
\end{aligned}$$

■

Hence the variational inequality (4.2) admits a unique solution. Moreover, in consequence of the alternative representation (4.4) of the right hand side g as defined in (3.11), we obtain the following corollary.

Corollary 4.2 *For any $u_0, \overline{u} \in L_2(\Omega)$ there hold the identities*

$$M_1\overline{u} = (-\frac{1}{2}I + K')V^{-1}M_0\overline{u}, \tag{4.5}$$

$$M_{11}u_0 = K_1'V^{-1}M_0u_0 + (\frac{1}{2}I - K')V^{-1}V_1V^{-1}M_0u_0 - (\frac{1}{2}I - K')V^{-1}M_{10}u_0. \tag{4.6}$$

4.1 Galerkin boundary element approximations

In what follows, we study the numerical solution of the variational inequality (4.2) by a Galerkin boundary element method. The ellipticity of the Schur complement boundary integral operator $\mathcal{T}_\alpha$ will imply the quasi–optimality of Galerkin approximations. Let us first introduce some finite dimensional trial spaces.

For the approximating subspaces of $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ it is customary to use tensor products of spaces of functions of the space variables and of spaces of functions of the time variable. We introduce a standard class of tensor product spaces

$$Q_h^{d_x,d_t}(\Sigma) = S_{h_x}^{d_x}(\Gamma) \otimes T_{h_t}^{d_t}$$

which are based on polynomials of degree d_t in time and polynomials of degree d_x in space, see [3, 9]. We choose an approximation for the Neumann data ω, q which is piecewise constant both in space and in time. For continuous functions z_1 and z_2, we define the discrete convex set

$$\mathcal{U}_h := \Big\{ w_h \in Q_h^{1,0}(\Sigma) \colon z_1(x_i,t_j) \le w_h(x_i,t_j) \le z_2(x_i,t_j) \quad \text{for all nodes } (x_i,t_j) \in \Sigma \Big\},$$

where $Q_h^{1,0}(\Sigma)$ is a boundary element space of piecewise linear and continuous basis functions in space and piecewise constant ones in time. Then the Galerkin discretization of the variational inequality (4.2) is to find $z_h \in \mathcal{U}_h$ such that

$$\langle \mathcal{T}_\alpha z_h, w_h - z_h \rangle_\Sigma \ge \langle g, w_h - z_h \rangle_\Sigma \quad \text{for all } w_h \in \mathcal{U}_h. \tag{4.7}$$

Theorem 4.3 *Let $z \in \mathcal{U}_{ad}$ and $z_h \in \mathcal{U}_h$ be the unique solutions of the variational inequalities* (4.2) *and* (4.7)*, respectively. Then there holds the error estimate*

$$\|z - z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \le c \left(h_x^{s-\frac{1}{2}} + h_t^{\frac{1}{2}(s-\frac{1}{2})} \right) \|z\|_{H^{s,\frac{s}{2}}(\Sigma)}, \tag{4.8}$$

when assuming $z, z_1, z_2 \in H^{s,\frac{s}{2}}(\Sigma)$ and $\mathcal{T}_\alpha z - g \in H^{s-1,\frac{s-1}{2}}(\Sigma)$ for some $s \in [\frac{1}{2}, 2]$.

Proof. The assertion follows from standard a priori error estimates for first kind variational inequalities, see in particular the discussion in [14]. ∎

Since the composed boundary integral operator $\mathcal{T}_\alpha$ and the right hand side g as defined in (4.3), (4.4) do not allow a practical implementation in general, instead of (4.7) we consider a perturbed variational inequality to find $\widehat{z}_h \in \mathcal{U}_h$ such that

$$\langle \widehat{\mathcal{T}}_\alpha \widehat{z}_h, w_h - \widehat{z}_h \rangle_\Sigma \ge \langle \widehat{g}, w_h - \widehat{z}_h \rangle_\Sigma \quad \text{for all } w_h \in \mathcal{U}_h. \tag{4.9}$$

Theorem 4.4 *Let $\widehat{\mathcal{T}}_\alpha : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ be a bounded and $Q_h^{1,0}(\Sigma)$–elliptic approximation of $\mathcal{T}_\alpha$ satisfying*

$$\langle \widehat{\mathcal{T}}_\alpha z_h, z_h\rangle_\Sigma \geq c_1^{\widehat{\mathcal{T}}_\alpha} \|z_h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z_h \in Q_h^{1,0}(\Sigma)$$

and

$$\|\widehat{\mathcal{T}}_\alpha z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_2^{\widehat{\mathcal{T}}_\alpha} \|z\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Let $\widehat{g} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ be some approximation of g. For the unique solution $\widehat{z}_h \in \mathcal{U}_h$ of the perturbed variational inequality (4.9) *there holds the error estimate*

$$\|z-\widehat{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c_1\|z-z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + c_2\Big(\|(\mathcal{T}_\alpha-\widehat{\mathcal{T}}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + \|g-\widehat{g}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}\Big), \quad (4.10)$$

where $z_h \in \mathcal{U}_h$ is the unique solution of the discrete variational inequality (4.7).

Proof. Since the operator $\widehat{\mathcal{T}}_\alpha$ is bounded and $Q_h^{1,0}(\Sigma)$–elliptic, the discrete variational inequality (4.9) admits a unique solution. From this we further obtain

$$\begin{aligned}
c_1^{\widehat{\mathcal{T}}_\alpha}\|z_h-\widehat{z}_h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} &\leq \langle \widehat{\mathcal{T}}_\alpha(z_h-\widehat{z}_h), z_h-\widehat{z}_h\rangle_\Sigma \\
&\leq \langle \widehat{\mathcal{T}}_\alpha z_h, z_h-\widehat{z}_h\rangle_\Sigma + \langle \widehat{g}, \widehat{z}_h - z_h\rangle_\Sigma + \langle \mathcal{T}_\alpha z_h, \widehat{z}_h - z_h\rangle_\Sigma - \langle g, \widehat{z}_h - z_h\rangle_\Sigma \\
&= \langle \widehat{\mathcal{T}}_\alpha z_h, z_h-\widehat{z}_h\rangle_\Sigma + \langle \widehat{g}-g, \widehat{z}_h - z_h\rangle_\Sigma + \langle \mathcal{T}_\alpha z_h, \widehat{z}_h - z_h\rangle_\Sigma \\
&\leq \Big(\|(\widehat{\mathcal{T}}_\alpha-\mathcal{T}_\alpha)z_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + \|g-\widehat{g}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}\Big)\|z_h-\widehat{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.
\end{aligned}$$

By using the triangle inequality and the boundedness of $\mathcal{T}_\alpha$ and $\widehat{\mathcal{T}}_\alpha$ we have

$$\begin{aligned}
\|(\widehat{\mathcal{T}}_\alpha-\mathcal{T}_\alpha)z_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} &\leq \|(\widehat{\mathcal{T}}_\alpha-\mathcal{T}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + \|(\mathcal{T}_\alpha-\widehat{\mathcal{T}}_\alpha)(z-z_h)\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \\
&\leq \|(\widehat{\mathcal{T}}_\alpha-\mathcal{T}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + (c_2^{\mathcal{T}_\alpha} + c_2^{\widehat{\mathcal{T}}_\alpha})\|z-z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.
\end{aligned}$$

The assertion now follows from the triangle inequality

$$\|z-\widehat{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq \|z-z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + \|z_h-\widehat{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

∎

It remains to define the appropriate approximations $\widehat{\mathcal{T}}_\alpha$, $\widehat{g}$ which are based on the use of boundary element methods. For $z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, the application of $\mathcal{T}_\alpha z$ reads

$$\mathcal{T}_\alpha z = \alpha\widetilde{D}z + \kappa_T D_1 z - \kappa_T K_1' \omega_z - \kappa_T(\frac{1}{2}I + K')q_z,$$

where $q_z, \omega_z \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ are the unique solutions of the boundary integral equations

$$V\omega_z = (\frac{1}{2}I + K)z, \quad Vq_z = K_1 z - V_1\omega_z.$$

Let $Q_h^{0,0}(\Sigma)$ be another boundary element space of piecewise constant basis functions both in space and in time. Let $q_{z,h} \in Q_h^{0,0}(\Sigma)$ be the unique solution of the Galerkin variational problem

$$\langle V q_{z,h}, \theta_h \rangle_\Sigma = \langle K_1 z - V_1 \omega_{z,h}, \theta_h \rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma),$$

where $\omega_{z,h} \in Q_h^{0,0}(\Sigma)$ solves

$$\langle V \omega_{z,h}, \theta_h \rangle_\Sigma = \langle (\frac{1}{2} I + K) z, \theta_h \rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma).$$

We are now in a position to define an approximation $\widehat{\mathcal{T}}_\alpha$ of the operator $\mathcal{T}_\alpha$ by

$$\widehat{\mathcal{T}}_\alpha z = \alpha \widetilde{D} z + \kappa_T D_1 z - \kappa_T K_1' \omega_{z,h} - \kappa_T (\frac{1}{2} I + K') q_{z,h}. \tag{4.11}$$

Lemma 4.5 *The approximate operator* $\widehat{\mathcal{T}}_\alpha : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ *as defined in* (4.11) *is bounded, i.e.,*

$$\|\widehat{\mathcal{T}}_\alpha z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_2^{\widehat{\mathcal{T}}_\alpha} \|z\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma),$$

and there holds the error estimate

$$\|(\mathcal{T}_\alpha - \widehat{\mathcal{T}}_\alpha) z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_1 \inf_{\theta_h \in Q_h^{0,0}(\Sigma)} \|q_z - \theta_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + c_2 \|\omega_z - \omega_{z,h}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}, \tag{4.12}$$

where $\mathcal{T}_\alpha$ *was defined in* (4.3).

Proof. The boundedness of the operator $\widehat{\mathcal{T}}_\alpha$ follows from the mapping properties of all boundary integral operators involved. In particular, the Galerkin boundary element solutions $\omega_{z,h}$, $q_{z,h}$ in (4.11) satisfy

$$\|\omega_{z,h}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_1 \|z\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}, \quad \|q_{z,h}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_2 \|z\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

For the error estimate (4.12) let $z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ be arbitrary but fixed. By definition, we have

$$\mathcal{T}_\alpha z = \alpha \widetilde{D} z + \kappa_T D_1 z - \kappa_T K_1' \omega_z - \kappa_T (\frac{1}{2} I + K') q_z,$$

where

$$V \omega_z = (\frac{1}{2} I + K) z, \quad V q_z = K_1 z - V_1 \omega_z.$$

By using (4.11), we then obtain

$$\mathcal{T}_\alpha z - \widehat{\mathcal{T}}_\alpha z = \kappa_T K_1' (\omega_{z,h} - \omega_z) + \kappa_T (\frac{1}{2} I + K') (q_{z,h} - q_z),$$

where $q_{z,h} \in Q_h^{0,0}(\Sigma)$ is the unique solution of the Galerkin variational problem

$$\langle V q_{z,h}, \theta_h \rangle_\Sigma = \langle K_1 z - V_1 \omega_{z,h}, \theta_h \rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma),$$

and $\omega_{z,h} \in Q_h^{0,0}(\Sigma)$ solves

$$\langle V\omega_{z,h}, \theta_h\rangle_\Sigma = \langle (\frac{1}{2}I + K)z, \theta_h\rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma).$$

Moreover we define $\widehat{q}_{z,h} \in Q_h^{0,0}(\Sigma)$ as the unique solution of the Galerkin variational problem

$$\langle V\widehat{q}_{z,h}, \theta_h\rangle_\Sigma = \langle K_1 z - V_1\omega_z, \theta_h\rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma).$$

Then the perturbed Galerkin orthogonality

$$\langle V(q_{z,h} - \widehat{q}_{z,h}), \theta_h\rangle_\Sigma = \langle V_1(\omega_{z,h} - \omega_z), \theta_h\rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma)$$

follows. This implies the inequality

$$\|q_{z,h} - \widehat{q}_{z,h}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \le \frac{1}{c_1^V}\|V_1(\omega_{z,h} - \omega_z)\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \le \frac{c_2^{V_1}}{c_1^V}\|\omega_{z,h} - \omega_z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}.$$

Therefore, by the boundedness of the operators $K', K_1' : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and by the triangle inequality we conclude

$$\begin{aligned}
\|(\mathcal{T}_\alpha - \widehat{\mathcal{T}}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} &\le c_2^{K_1'}\|\omega_{z,h} - \omega_z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + c_2^{K'}\|q_{z,h} - q_z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \\
&\le c_2^{K_1'}\|\omega_{z,h} - \omega_z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + c_2^{K'}\|q_{z,h} - \widehat{q}_{z,h}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + c_2^{K'}\|\widehat{q}_{z,h} - q_z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}.
\end{aligned}$$

The assertion now follows by applying Cea's lemma. ■

By using the approximation property of the trial space $Q_h^{0,0}(\Sigma)$, we conclude an error estimate from (4.12) when assuming some regularity of q_z and ω_z, respectively.

Corollary 4.6 *Assume $q_z, \omega_z \in H^{s,\frac{s}{2}}(\Sigma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|(\mathcal{T}_\alpha - \widehat{\mathcal{T}}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \le c\left(h_x^{\frac{1}{2}} + h_t^{\frac{1}{4}}\right)\left(h_x^s + h_t^{\frac{s}{2}}\right)\left(\|q_z\|_{H^{s,\frac{s}{2}}(\Sigma)} + \|\omega_z\|_{H^{s,\frac{s}{2}}(\Sigma)}\right). \qquad (4.13)$$

Similarly, the right hand side in (4.4) can be rewritten as

$$g = \kappa_T(\frac{1}{2}I + K')q_{\overline{u},u_0} + \kappa_T K_1'\omega_{u_0} - \kappa_T M_1\overline{u} + \kappa_T M_{11}u_0,$$

where $q_{\overline{u},u_0} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is the unique solution of the boundary integral equation

$$(Vq_{\overline{u},u_0})(x,t) = (M_0\overline{u})(x,t) - (M_{10}u_0)(x,t) - (V_1\omega_{u_0})(x,t) \quad \text{for } (x,t) \in \Sigma,$$

and $\omega_{u_0} \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ solves

$$(V\omega_{u_0})(x,t) = -(M_0u_0)(x,t) \quad \text{for } (x,t) \in \Sigma.$$

Hence we define approximate Galerkin solutions $\widetilde{q}_h, \widetilde{\omega}_h \in Q_h^{0,0}(\Sigma)$ of $q_{\overline{u},u_0}$ and ω_{u_0}, and then we can introduce the approximation

$$\widehat{g} = \kappa_T(\frac{1}{2}I + K')\widetilde{q}_h + \kappa_T K_1'\widetilde{\omega}_h - \kappa_T M_1\overline{u} + \kappa_T M_{11}u_0 \tag{4.14}$$

and we obtain the error estimate

$$\|g - \widehat{g}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c\left(h_x^{\frac{1}{2}} + h_t^{\frac{1}{4}}\right)\left(h_x^s + h_t^{\frac{s}{2}}\right)\left(\|q_{\overline{u},u_0}\|_{H^{s,\frac{s}{2}}(\Sigma)} + \|\omega_{u_0}\|_{H^{s,\frac{s}{2}}(\Sigma)}\right), \tag{4.15}$$

when assuming $q_{\overline{u},u_0}, \omega_{u_0} \in H^{s,\frac{s}{2}}(\Sigma)$ for some $s \in [0,1]$.

4.2 Approximate variational inequality

By using the approximations (4.11) and (4.14), the perturbed variational inequality (4.9) reads to find $\widehat{z}_h \in \mathcal{U}_h$ such that

$$\begin{aligned}&\langle \alpha\widetilde{D}\widehat{z}_h + \kappa_T D_1\widehat{z}_h - \kappa_T K_1'\omega_{\widehat{z}_h,h} - \kappa_T(\frac{1}{2}I + K')q_{\widehat{z}_h,h}, w_h - \widehat{z}_h\rangle_\Sigma \\ &\quad \geq \langle \kappa_T(\frac{1}{2}I + K')\widetilde{q}_h + \kappa_T K_1'\widetilde{\omega}_h - \kappa_T M_1\overline{u} + \kappa_T M_{11}u_0, w_h - \widehat{z}_h\rangle_\Sigma \quad \text{for all } w_h \in \mathcal{U}_h\end{aligned}$$

which can be written as

$$\begin{aligned}\langle \alpha\widetilde{D}\widehat{z}_h + \kappa_T D_1\widehat{z}_h - \kappa_T K_1'\omega_h - \kappa_T(\frac{1}{2}I + K')q_h, w_h - \widehat{z}_h\rangle_\Sigma& \\ \geq \langle \kappa_T M_{11}u_0 - \kappa_T M_1\overline{u}, w_h - \widehat{z}_h\rangle_\Sigma&\end{aligned} \tag{4.16}$$

for all $w_h \in \mathcal{U}_h$, where we introduce $q_h := q_{\widehat{z}_h,h} + \widetilde{q}_h \in Q_h^{0,0}(\Sigma)$ which is the unique solution of the Galerkin variational problem

$$\langle Vq_h, \theta_h\rangle_\Sigma = \langle K_1\widehat{z}_h - V_1\omega_h, \theta_h\rangle_\Sigma + \langle M_0\overline{u} - M_{10}u_0, \theta_h\rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma), \tag{4.17}$$

and $\omega_h := \omega_{\widehat{z}_h,h} + \widetilde{\omega}_h \in Q_h^{0,0}(\Sigma)$ which solves

$$\langle V\omega_h, \theta_h\rangle_\Sigma = \langle(\frac{1}{2}I + K)\widehat{z}_h - M_0u_0, \theta_h\rangle_\Sigma \quad \text{for all } \theta_h \in Q_h^{0,0}(\Sigma), \tag{4.18}$$

see the corresponding boundary integral equations (3.7), (3.1).

Let

$$\omega_h(x,t) = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N_0-1} \omega_{\ell k}\varphi_\ell^0(x)\psi_k^0(t), \quad q_h(x,t) = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N_0-1} q_{\ell k}\varphi_\ell^0(x)\psi_k^0(t),$$

and

$$\widehat{z}_h(x,t) = \sum_{k=0}^{N-1}\sum_{n=0}^{N_1-1} z_{nk}\varphi_n^1(x)\psi_k^0(t), \quad (x,t) \in \Sigma,$$

where N_i denotes the dimension of $S^i_{h_x}(\Gamma)$, $i=0,1$ and N is the number of time steps. Substituting these expansions into (4.17) with the test functions $\theta_h(x,t)=\varphi^0_i(x)\kappa_T\psi^0_j(t)$ for $i=0,1,\dots,N_0-1$; $j=0,1,\dots,N-1$, we get

$$\sum_{k=0}^{N-1}\sum_{\ell=0}^{N_0-1}\Big(\omega_{\ell k}\langle V_1[\varphi^0_\ell(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma+q_{\ell k}\langle V[\varphi^0_\ell(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma\Big)$$
$$-\sum_{k=0}^{N-1}\sum_{n=0}^{N_1-1}z_{nk}\langle K_1[\varphi^1_n(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma=\langle M_0\overline{u}-M_{10}u_0,\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma$$

for all $i=0,1,\dots,N_0-1$; $j=0,1,\dots,N-1$.

Since the last equation is indexed by four integers, it requires some ordering or partitioning of the unknowns. For $0\le k\le N-1$ we define vectors $\underline{\omega}_k$, $\underline{q}_k\in\mathbb{R}^{N_0}$ and $\underline{z}_k\in\mathbb{R}^{N_1}$ by

$$\omega_k[\ell]=\omega_{\ell k},\quad q_k[\ell]=q_{\ell k},\quad z_k[n]=z_{nk}\quad\text{for}\quad \ell=0,1,\dots,N_0-1;n=0,1,\dots,N_1-1.$$

Similarly, $\underline{f}^1_j$ denote vectors of length N_0 whose components are given by

$$f^1_j[i]=\langle M_0\overline{u}-M_{10}u_0,\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma,\quad\text{for}\quad i=0,1,\dots,N_0-1;j=0,1,\dots,N-1.$$

Finally, we define matrices V^1_{jk}, $V_{jk}\in\mathbb{R}^{N_0\times N_0}$ and $K^1_{jk}\in\mathbb{R}^{N_0\times N_1}$ for $0\le k,j\le N-1$ by

$$\begin{aligned}V^1_{jk}[i][\ell]&=\langle V_1[\varphi^0_\ell(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma,\\ V_{jk}[i][\ell]&=\langle V[\varphi^0_\ell(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma,\\ K^1_{jk}[i][n]&=\langle K_1[\varphi^1_n(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma,\end{aligned}$$

for $i,\ell=0,1,\dots,N_0-1$; $n=0,1,\dots,N_1-1$.

With these notations, the system (4.17) can be written in the form

$$\sum_{k=0}^{N-1}\Big(V^1_{jk}\underline{\omega}_k+V_{jk}\underline{q}_k-K^1_{jk}\underline{z}_k\Big)=\underline{f}^1_j,\quad\text{for}\quad j=0,1,\dots,N-1.\tag{4.19}$$

In the same way, the system (4.18) reads

$$\sum_{k=0}^{N-1}V_{jk}\underline{\omega}_k-\sum_{k=0}^{N-1}\left(\frac{1}{2}M_{jk}+K_{jk}\right)\underline{z}_k=\underline{f}^2_j,\quad\text{for}\quad j=0,1,\dots,N-1,\tag{4.20}$$

where the matrices $M_{jk},K_{jk}\in\mathbb{R}^{N_0\times N_1}$ are defined by

$$\begin{aligned}M_{jk}[i][n]&=\langle\varphi^1_n(x)\psi^0_k(t),\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma,\\ K_{jk}[i][n]&=\langle K[\varphi^1_n(x)\psi^0_k(t)],\varphi^0_i(x)\kappa_T\psi^0_j(t)\rangle_\Sigma\end{aligned}$$

and

$$f_j^2[i] = -\langle M_0 u_0, \varphi_i^0(x)\kappa_T \psi_j^0(t)\rangle_\Sigma.$$

Let us rewrite the linear systems (4.19) and (4.20) as follows. For the N^2 matrices A_{jk}, $j,k = 0,1,\dots,N-1$, which correspond to one of the layer heat potentials A, i.e.,

$$A_{jk}[i][\ell] = \langle A[\varphi_\ell^0(x)\psi_k^0(t)], \varphi_i^0(x)\kappa_T\psi_j^0(t)\rangle_\Sigma,$$

we denote a block matrix A_h by

$$A_h := \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0,N-1} \\ A_{10} & A_{11} & \cdots & A_{1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N-1,0} & A_{N-1,1} & \cdots & A_{N-1,N-1} \end{pmatrix}.$$

We define a vector $\underline{a}$ which is constructed from the N vectors $\underline{a}_k$ by

$$\underline{a} := \begin{pmatrix} \underline{a}_0^\top & \underline{a}_1^\top & \cdots & \underline{a}_{N-1}^\top \end{pmatrix}^\top.$$

With these notations, the inner product of two vectors $A_h\underline{a}$ and $\underline{b}$ can be expressed by the time–twisted duality, i.e.,

$$(A_h\underline{a}, \underline{b}) = \langle A a_h, \kappa_T b_h\rangle_\Sigma,$$

where a_h, b_h are trial functions whose coefficients of the expansions in trial spaces correspond to the vectors $\underline{a}, \underline{b}$. Here the operator A can be one of the layer heat potentials $V, K, V_1, \dots$. In case of the identity operator, we have a mass matrix M_h, as usual.

We now rewrite the linear systems (4.19) and (4.20) in the forms

$$V_{1,h}\underline{\omega} + V_h\underline{q} - K_{1,h}\underline{z} = \underline{f}^1 \tag{4.21}$$

and

$$V_h\underline{\omega} - (\frac{1}{2}M_h + K_h)\underline{z} = \underline{f}^2, \tag{4.22}$$

respectively.

4.3 Discrete variational inequality

Analogously, we can also reformulate the perturbed variational inequality (4.16) to find $\underline{z} \in \mathbb{R}^{N_1 N} \leftrightarrow \widehat{z}_h \in \mathcal{U}_h$ such that

$$(\alpha\widetilde{D}_h\underline{z} + D_{1,h}\underline{z} - K_{1,h}^\top\underline{\omega} - (\frac{1}{2}M_h^\top + K_h^\top)\underline{q}, \underline{w} - \underline{z}) \geq (\underline{f}^3, \underline{w} - \underline{z}) \tag{4.23}$$

for all $\underline{w} \in \mathbb{R}^{N_1 N} \leftrightarrow w_h \in \mathcal{U}_h$, where $\underline{\omega}, \underline{q} \in \mathbb{R}^{N_0 N}$ are the unique solutions of the linear systems (4.22), (4.21), respectively. Here, in addition, we define the matrices $D_{jk}^1, \widetilde{D}_{jk} \in$

$\mathbb{R}^{N_1 \times N_1}$ and the vectors $\underline{f}_j^3 \in \mathbb{R}^{N_1}$ by

$$\begin{aligned}
D^1_{jk}[m][n] &= \langle D_1[\varphi^1_n(x)\psi^0_k(t)], \varphi^1_m(x)\kappa_T\psi^0_j(t)\rangle_\Sigma, \\
\widetilde{D}_{jk}[m][n] &= \frac{1}{2}\langle D[\varphi^1_n(x)\psi^0_k(t)], \varphi^1_m(x)\psi^0_j(t)\rangle_\Sigma + \frac{1}{2}\langle D[\varphi^1_n(x)\kappa_T\psi^0_k(t)], \varphi^1_m(x)\kappa_T\psi^0_j(t)\rangle_\Sigma, \\
f^3_j[m] &= \langle M_{11}u_0 - M_1\overline{u}, \varphi^1_m(x)\kappa_T\psi^0_j(t)\rangle_\Sigma
\end{aligned}$$

for $j, k = 0, 1, \ldots, N-1$; $m, n = 0, 1, \ldots, N_1 - 1$. Note that

$$(\widetilde{D}_h\underline{a}, \underline{b}) = \langle \widetilde{D}a_h, b_h\rangle_\Sigma \quad \text{for all } \underline{a}, \underline{b} \in \mathbb{R}^{N_1 N} \leftrightarrow a_h, b_h \in Q_h^{1,0}(\Sigma).$$

The Galerkin matrix V_h of the single layer heat potential V is symmetric and positive definite, hence it is invertible, and we can determine $\underline{\omega}$ and $\underline{q}$ from (4.22) and (4.21). Then the discrete variational inequality (4.23) is equivalent to

$$(\mathcal{T}_{\alpha,h}\underline{z}, \underline{w} - \underline{z}) \geq (\underline{g}, \underline{w} - \underline{z}) \quad \text{for all } \underline{w} \in \mathbb{R}^{N_1 N} \leftrightarrow w_h \in \mathcal{U}_h, \tag{4.24}$$

where

$$\begin{aligned}
\mathcal{T}_{\alpha,h} &= \alpha\widetilde{D}_h + D_{1,h} - (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}K_{1,h} - K_{1,h}^\top V_h^{-1}(\frac{1}{2}M_h + K_h) \qquad (4.25)\\
&\quad +(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h)
\end{aligned}$$

defines a symmetric Galerkin boundary element approximation of the self–adjoint operator $\mathcal{T}_\alpha$ and

$$\underline{g} = \underline{f}^3 + K_{1,h}^\top V_h^{-1}\underline{f}^2 + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\underline{f}^1 - V_{1,h}V_h^{-1}\underline{f}^2) \tag{4.26}$$

is the related boundary element approximation of the right hand side g as defined in (4.4).

Lemma 4.7 *The symmetric matrix $\mathcal{T}_{\alpha,h}$ as defined in* (4.25) *is positive definite, i.e.,*

$$(\mathcal{T}_{\alpha,h}\underline{z}, \underline{z}) \geq \alpha c_1^D \, \|z_h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } \underline{z} \in \mathbb{R}^{N_1 N} \leftrightarrow z_h \in Q_h^{1,0}(\Sigma).$$

Proof. While the symmetry of $\mathcal{T}_{\alpha,h}$ is obvious, the positive definiteness follows by using Lemma 3.3. Indeed, by using the symmetry of V_h and with $\underline{\omega} = V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{z}$, we have

$$\begin{aligned}
(\mathcal{T}_{\alpha,h}\underline{z}, \underline{z}) &= \alpha(\widetilde{D}_h\underline{z}, \underline{z}) + (D_{1,h}\underline{z}, \underline{z}) - 2(K_{1,h}\underline{z}, \underline{\omega}) + (V_{1,h}\underline{\omega}, \underline{\omega}), \\
&= \alpha\langle \widetilde{D}z_h, z_h\rangle_\Sigma + \langle D_1 z_h, \kappa_T z_h\rangle_\Sigma - 2\langle K_1 z_h, \kappa_T\omega_h\rangle_\Sigma + \langle V_1\omega_h, \kappa_T\omega_h\rangle_\Sigma, \\
&\geq \alpha\langle \widetilde{D}z_h, z_h\rangle_\Sigma \geq \alpha c_1^D \, \|z_h\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.
\end{aligned}$$

■

Hence we conclude the unique solvability of the variational inequality (4.24) and (4.9) as well. Moreover, we can derive an error estimate for the approximate control solution $\widehat{z}_h$ by applying Theorem 4.4 and with the error estimates (4.8), (4.13) and (4.15).

Theorem 4.8 *Let z and $\widehat{z}_h$ be the unique solutions of the variational inequalities* (4.2) *and* (4.9)*, respectively. Then there holds the error estimate*

$$\|z-\widehat{z}_h\|_{H^{\frac12,\frac14}(\Sigma)} \le c_1\Big(h_x^{s+\frac12}+h_t^{\frac12(s+\frac12)}\Big)\|z\|_{H^{s+1,\frac{s+1}{2}}(\Sigma)} + c_2\Big(h_x^{\frac12}+h_t^{\frac14}\Big)\Big(h_x^s+h_t^{\frac{s}{2}}\Big)\|q_z\|_{H^{s,\frac{s}{2}}(\Sigma)}$$
$$+c_3\Big(h_x^{\frac12}+h_t^{\frac14}\Big)\Big(h_x^s+h_t^{\frac{s}{2}}\Big)\Big(\|\omega_z\|_{H^{s,\frac{s}{2}}(\Sigma)}+\|q_{\overline{u},u_0}\|_{H^{s,\frac{s}{2}}(\Sigma)}+\|\omega_{u_0}\|_{H^{s,\frac{s}{2}}(\Sigma)}\Big)$$

when assuming $z \in H^{s+1,\frac{s+1}{2}}(\Sigma)$ and $q_z, \omega_z, q_{\overline{u},u_0}, \omega_{u_0} \in H^{s,\frac{s}{2}}(\Sigma)$ for some $s \in [0,1]$.

In particular, if there are constants $c_1, c_2 > 0$ such that

$$c_1 h_x^2 \le h_t \le c_2 h_x^2,$$

we obtain the estimate

$$\|z-\widehat{z}_h\|_{H^{\frac12,\frac14}(\Sigma)} \le c(z,\overline{u},u_0)\, h_x^{s+\frac12} \quad \text{for } z \in H^{s+1,\frac{s+1}{2}}(\Sigma), s\in[0,1]. \tag{4.27}$$

In the case of a non–constrained minimization problem, instead of the discrete variational inequality (4.24) we have to solve the linear system

$$\mathcal{T}_{\alpha,h}\underline{z} = \underline{g},$$

which can be written as

$$\begin{pmatrix} V_{1,h} & V_h & -K_{1,h} \\ V_h & & -(\frac12 M_h + K_h) \\ -K_{1,h}^\top & -(\frac12 M_h^\top + K_h^\top) & D_{1,h}+\alpha\widetilde{D}_h \end{pmatrix} \begin{pmatrix} \underline{\omega} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{f}^1 \\ \underline{f}^2 \\ \underline{f}^3 \end{pmatrix}. \tag{4.28}$$

5 Numerical results

As a numerical example we consider the Dirichlet boundary control problem (2.1)–(2.3) for a circular domain $\Omega = B_{0.5}(0) \subset \mathbb{R}^2$. For the boundary element discretization we use a uniform boundary element mesh on several levels L, with $N_0 = N_1 = 2^{L+2}$ nodes; and a uniform decomposition of the interval $(0,T)$ by N time steps. We use the trial space $Q_h^{1,0}(\Sigma)$ of piecewise linear and continuous basis functions in the space variable $x \in \Gamma$, and piecewise constant ones in the time variable $t \in (0,T)$ to approximate the Dirichlet control z. For the fluxes ω, q, we use the trial space $Q_h^{0,0}(\Sigma)$ of piecewise constant basis functions both in space and in time. We set $T = 0.5$ and

$$\overline{u}(x) = (x_1^2+x_2^2)\log(x_1^2+x_2^2) + 4x_1x_2, \quad u_0(x) = 0, \quad \alpha = 0.1, \quad z_1 = -1, \quad z_2 = 0.11.$$

Since the minimizer of (2.1) is not known, we use the boundary element solutions $z_{\mathrm{ref}}, \omega_{\mathrm{ref}}$ for $N_0 = N_1 = 512$ and $N = 512$ as reference solutions. In Table 1, we present the errors for the control z and the estimated order of convergence (eoc). The errors of the flux ω in the $L_2(\Sigma)$ norm are also given. Since the data are smooth in this case, we expect the optimal order of convergence 1.5 for the control z in the energy space $H^{\frac12,\frac14}(\Sigma)$ which agrees with the theoretical results, see (4.27). In addition, in Fig. 1 we present the final optimal solution $u(\cdot,T)$ with the given target function $\overline{u}$.

N_0, N_1	N	$\|\widehat{z}_h - z_{\text{ref}}\|_{L_2(\Sigma)}$	eoc	$\|\widehat{z}_h - z_{\text{ref}}\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}$	eoc	$\|\omega_h - \omega_{\text{ref}}\|_{L_2(\Sigma)}$	eoc
32	4	0.154249	-	1.050200	-	0.727606	-
64	16	0.054260	1.507	0.561425	0.903	0.321477	1.178
128	64	0.012887	2.074	0.212840	1.399	0.187099	0.781
256	256	0.003018	2.094	0.071736	1.569	0.064487	1.537
expected			2.0		1.5		1.0

Table 1: Errors and estimated order of convergence.

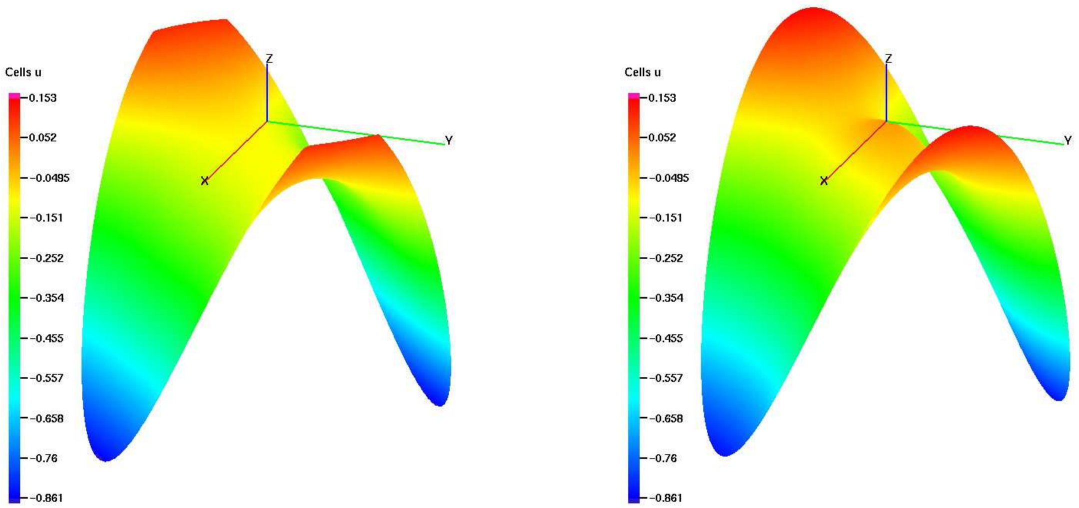

Figure 1: Final optimal solution (left) and target function (right).

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References

[1] R. A. Adams: Sobolev Spaces. Academic Press, New York, London, 1975.

[2] D. N. Arnold, P. J. Noon: Boundary integral equations of the first kind of the heat equation. In: Boundary Elements IX (C. A. Brebbia, G. Kuhn, W. L. Wendland eds.), Springer, Berlin, vol. 3, pp. 213–229, 1987.

[3] M. Costabel: Boundary integral operators for the heat equation. Integr. Equ. Oper. Theory 13 (1990) 498–552.

[4] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich: Optimization with PDE Constraints. Mathematical Modelling: Theory and Applications, vol. 23, Springer, Heidelberg, 2009.

[5] G. C. Hsiao, W. L. Wendland: Boundary Integral Equations. Springer, Heidelberg, 2008.

[6] K. Kunisch, B. Vexler: Constrained Dirichlet boundary control in L^2 for a class of evolution equations. SIAM J. Control Optim. 46 (2007) 1726–1753.

[7] J. L. Lions: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin, Heidelberg, New York, 1971.

[8] W. McLean: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[9] P. J. Noon: The single layer heat potential and Galerkin boundary element methods for the heat equation. Dissertation, The University of Maryland, 1988.

[10] G. Of, T. X. Phan, O. Steinbach: Boundary element methods for Dirichlet boundary control problems. Math. Meth. Appl. Sci., 33: 2187-2205, 2010.

[11] T. X. Phan: Boundary element methods for boundary control problems. PhD Thesis, TU Graz, 2011.

[12] G. Schmidt, B. N. Khoromskij: Boundary integral equations for the biharmonic Dirichlet problem on nonsmooth domains. J. Integr. Equ. Appl. 11 (1999) 217–253.

[13] O. Steinbach: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[14] O. Steinbach: Boundary element methods for variational inequalities. Berichte aus dem Institut für Numerische Mathematik, Bericht 2012/4, TU Graz, 2012.

[15] F. Troeltzsch: Optimale Steuerung partieller Differentialgleichungen. Theorie, Verfahren, Anwendungen. Vieweg, Braunschweig, 2005.

[16] R. Winther: Error estimates for a Galerkin approximation of a parabolic control problem. Ann. Mat. Pura Appl. 107 (1978) 173–206.

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